Lecture 1, Jan 9, 2023

Maxwell's Equations

Equation

Maxwell's Equations:

$$\text{Faraday's Law: } \vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\text{Ampere's Law: } \vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\text{Gauss's Law (Electric): } \vec{\nabla} \cdot \vec{E} = \frac{\rho_\nu}{\varepsilon}$$

$$\text{Gauss's Law (Magnetic): } \vec{\nabla} \cdot \vec{H} = 0$$

In a static field:

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_\nu}{\varepsilon}$$

$$\vec{\nabla} \times \vec{H} = 0$$

- In a static field, electric and magnetic fields are now independent
- Maxwell added the second term in Ampere's Law, connecting electric and magnetic fields

Electrostatics – The Beginning

- The *triboelectric series* ranks the tendency for different materials to gain or lose electrons
- Coulomb noticed the properties of the electric force (Coulomb's Law)
  - $|\vec{F}_e|$ dependent on $Q_1 Q_2$
  - $|\vec{F}_e| \propto \frac{1}{R^2}$
  - The direction of $\vec{F}_e$ acts along the line connecting Q_1 and Q_2
  - Like charges repel, opposite charges attract
- Mathematically we express this as $|\vec{F}_e| = F_e \propto \frac{Q_1 Q_2}{R^2}$

Definition

Coulomb's Law (scalar form):

$$F_e = k\frac{Q_1 Q_2}{R^2} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$

where $k = 9 \times 10^9\,\mathrm{Nm^2/C^2}$, $\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}$ is the permittivity of free space

Lecture 2, Jan 11, 2023

The Electric Field

- Expressing everything in terms of forces is cumbersome; using "fields" can simplify this
- While forces can be directly experienced, fields remove the immediate effects

  - An electric force requires two object, a source charge (the thing causing the force), and a test charge (the thing experiencing the force)
  - A field however only requires the source charge and fully characterizes it
- We can define the *electric field intensity* as $\vec{E}_{12} = \lim_{Q_2 \to 0} \frac{\vec{F}_{12}}{Q_{12}} = \lim_{Q_2 \to 0} \frac{1}{Q_{12}} k \frac{Q_1 Q_2}{R^2} \hat{a}_{12} = k \frac{Q_1}{R^2} \hat{a}_{12}$
  - The electric field has units of [N/C] = [V/m]
  - Now we have $\vec{F}_{12} = Q_2 \vec{E}_{12}$

Definition

The *electric field* caused by a charge Q_1 is defined as

$$\vec{E}_{12} = k \frac{Q_1}{R^2} \hat{a}_{12} = \frac{Q_1}{4\pi\varepsilon_0 R^2} \hat{a}_{12}$$

The electric field has 4 key properties:

1. $\vec{E}$ points away from positive charges
2. $\vec{E}$ points towards negative charges
3. $\vec{E}$ points along the line connecting the source point to the measurement point
4. $\vec{E}$ is linear, so we can superimpose electric fields from multiple charges

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The electric field at point $\vec{R}$ due to a point charge at $\vec{R}'$ is

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'\|^2} \hat{a}_{12} = \frac{Q_1}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'\|^3} (\vec{R} - \vec{R}')$$

- The first 2 properties are by convention – we always think of the electric field coming from a positive charge and going into a negative charge somewhere

Position Vectors

- In Cartesian coordinates, a point $P(x, y, z)$ is specified by a position vector $\vec{R} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ where $\hat{a}_x, \hat{a}_y, \hat{a}_z$ are the 3 unit vectors
- In cylindrical coordinates, a point is specified by $P(r, \phi, z)$; unit vectors are $\hat{a}_r, \hat{a}_\phi, \hat{a}_z$
  - $\hat{a}_z$ is constant, but $\hat{a}_r, \hat{a}_\phi$ change based on angle!
  - A position vector is described by $\vec{R} = r\hat{a}_r + z\hat{a}_z$, because ϕ is encoded in $\hat{a}_r$
    * In Cartesian coordinates $\hat{a}_r = \cos(\phi)\hat{a}_x + \sin(\phi)\hat{a}_y$
- In spherical coordinates, a point is specified by $P(R, \theta, \phi)$ (**note ϕ is the angle in the xy plane!**); unit vectors are $\hat{a}_R, \hat{a}_\phi, \hat{a}_\theta$
  - All unit vectors change based on where you are
  - A position vector is described by $\vec{R} = R\hat{a}_r$
  - $\hat{a}_r = \sin\theta\cos\phi\hat{a}_x + \sin\theta\sin\phi\hat{a}_y + \cos\theta\hat{a}_z$

Lecture 3, Jan 13, 2023

Continuous Charge Distributions

- What is the electric field due to a charged plate?
  - Consider a point at $P(0, 0, z)$, and a plate with total charge Q and area A on the xy plane
  - Break the plate into pieces, by superposition $\vec{E}_{tot} = \sum_{i=1}^{N} \vec{E}_i = \sum_{i=1}^{N} \frac{Q_i(\vec{R} - \vec{R}'_i)}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'_i\|^2}$
  - As $N \to \infty$, the summation becomes an integral and Q_i become $\mathrm{d}Q'$, which are point charges
    * Note primes denote source charge

- Define the *charge density* ρ_s, in this case with area C/m^2 and $\rho_s = \frac{Q}{A}$
- $\vec{E}_{tot} = \iint_S \frac{(\vec{R} - \vec{R}')}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'\|^3} \,\mathrm{d}Q'$
  - $\vec{R}'$ is a function of x and y
- Instead of considering discrete (point charges), which are confined to an infinitesimally small point, in most practical problems charge is distributed in one or more dimensions

Definition

There are 3 types of *continuous charge distributions*:

- Linear: $Q = \int \rho_l \,\mathrm{d}l$
  $$\rho_l = \frac{Q}{L}$$
- Surface: $Q = \iint_S \rho_s \,\mathrm{d}S$
  $$\rho_s = \frac{Q}{A}$$
- Volume: $Q = \iiint_V \rho_v \,\mathrm{d}V$
  $$\rho_v = \frac{Q}{V}$$

In each case ρ denotes the charge density, and the subscript denotes the dimensionality

Differential Elements in Orthogonal Coordinate Systems

- In Cartesian coordinates:
  - Differential lengths are $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$, so a differential length vector is $\mathrm{d}\vec{l} = \mathrm{d}x\,\hat{a}_x + \mathrm{d}y\,\hat{a}_y + \mathrm{d}x\,\hat{a}_z$
  - Differential surface vectors are $\begin{cases} \mathrm{d}\vec{s}_x = \mathrm{d}y\,\mathrm{d}z\,\hat{a}_x \\ \mathrm{d}\vec{s}_y = \mathrm{d}x\,\mathrm{d}z\,\hat{a}_y \\ \mathrm{d}\vec{s}_z = \mathrm{d}x\,\mathrm{d}y\,\hat{a}_z \end{cases}$
  - Differential volume is $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
- In cylindrical coordinates:
  - Differential lengths are $\mathrm{d}r, r\,\mathrm{d}\phi, \mathrm{d}z$
  - Differential length vector is $\mathrm{d}\vec{l} = \mathrm{d}r\,\hat{a}_r + r\,\mathrm{d}\phi\,\hat{a}_\phi + \mathrm{d}z\,\hat{a}_z$
  - Differential surface vectors are $\begin{cases} \mathrm{d}\vec{s}_r = r\,\mathrm{d}\phi\,\mathrm{d}z\,\hat{a}_r \\ \mathrm{d}\vec{s}_\phi = \mathrm{d}r\,\mathrm{d}z\,\hat{a}_\phi \\ \mathrm{d}\vec{s}_z = r\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{a}_z \end{cases}$
    - $\mathrm{d}\vec{s}_r$ represents the cylindrical wall
    - $\mathrm{d}\vec{s}_\phi$ represents a vertical plane coming out of the z axis
    - $\mathrm{d}\vec{s}_z$ represents a horizontal plane
  - Differential volume is $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- In spherical coordinates:
  - Differential lengths are $\mathrm{d}R, R\,\mathrm{d}\theta, R\sin\theta\,\mathrm{d}\phi$
  - Differential length vector is $\mathrm{d}\vec{l} = \mathrm{d}R\,\hat{a}_R + R\sin\theta\,\mathrm{d}\phi\,\hat{a}_\phi + R\,\mathrm{d}\theta\,\hat{a}_\theta$
  - Differential surface vectors are $\begin{cases} \mathrm{d}\vec{s}_R = R^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{a}_R \\ \mathrm{d}\vec{s}_\phi = R\,\mathrm{d}\theta\,\mathrm{d}R\,\hat{a}_\phi \\ \mathrm{d}\vec{s}_\theta = R\sin\theta\,\mathrm{d}\phi\,\mathrm{d}R\,\hat{a}_\theta \end{cases}$
  - Differential volume is $\mathrm{d}V = R^2\sin\theta\,\mathrm{d}R\,\mathrm{d}\phi\,\mathrm{d}\theta$

Lecture 4, Jan 16, 2023

Example: Electric Field Above a Charged Disk

- Charged disk of radius a with total charge Q, measured at a point $P(0,0,h)$
  1. $\vec{E}_{tot} = \iint \mathrm{d}\vec{E} = \iint_S \frac{(\vec{R} - \vec{R}')}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'\|^2} \,\mathrm{d}Q'$
  2. $\mathrm{d}Q'$ is the differential charge, $\rho_s \,\mathrm{d}s'$; in this case $\mathrm{d}s = \mathrm{d}s_z$ so $\mathrm{d}Q' = \rho_s r' \,\mathrm{d}\phi' \mathrm{d}r' = \frac{Q}{\pi a^2} r' \,\mathrm{d}\phi' \mathrm{d}r'$
  3. $\vec{R} = h\hat{a}_z$; $\vec{R}' = r'\hat{a}_r$ so $\vec{R} - \vec{R}' = -r'\hat{a}_r + h\hat{a}_z = -r'\cos\phi'\hat{a}_x - r'\sin\phi'\hat{a}_y + h\hat{a}_z$
  4. $\iint \mathrm{d}\vec{E} = \int_0^a \int_0^{2\pi} \frac{\frac{Q}{\pi a^2} r'}{4\pi\varepsilon_0 \left((r')^2 + h^2\right)^{\frac{3}{2}}} (-r'\cos\phi'\hat{a}_x - r'\sin\phi'\hat{a}_y + h\hat{a}_z) \mathrm{d}\phi' \mathrm{d}r'$
  5. The disk is symmetric about the z axis, so there will only be a z component in the total field
  6. $\iint_S \mathrm{d}\vec{E} = \frac{Qh\hat{a}_z}{4\pi(\pi a^2)\varepsilon_0} \int_0^a \int_0^{2\pi} \frac{r'}{((r')^2 + h^2)^{\frac{3}{2}}} \,\mathrm{d}\phi' \mathrm{d}r' = \frac{\rho_s}{2\varepsilon_0} \left(\frac{h}{|h|} - \frac{h}{\sqrt{a^2 + h}} \right) \hat{a}_z$
- In general the steps are:
  1. Select a coordinate system
  2. Find $\mathrm{d}Q'$
  3. Find $\vec{R}, \vec{R}'$ and $\vec{R} - \vec{R}'$
  4. Write out the integral
  5. Look for symmetry before evaluating the integral!
  6. Integrate
- Note if we take $r \to \infty$ we effectively have an infinite plate of charge, then we get $\vec{E}_{tot} = \pm\frac{\rho_s}{2\varepsilon_0}\hat{a}_z$

Lecture 5, Jan 18, 2023

Lecture 6, Jan 20, 2023

Gauss's Law

Equation

Electrostatics is governed by two fundamental postulates:
In differential form:

$$\vec{\nabla} \times \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

In integral form:

$$\oint_C \vec{E} \cdot \mathrm{d}\vec{l} = 0$$
$$\oiint_S \vec{D} \cdot \mathrm{d}\vec{S} = Q_{enc}$$

- Electrostatics deal with systems with stationary charges; we represent the field with $\vec{E}$ or $\vec{D}$
  - $\vec{E}$ is the electric field density with units of $\mathrm{V/m} = \mathrm{N/C}$
  - $\vec{D}$ is the electric flux density with units of $\mathrm{C/m^2}$
- $\vec{E}$ and $\vec{D}$ are related by the parameter ε: $\vec{D} = \varepsilon_r\varepsilon_0 = \varepsilon\vec{E}$
  - ε is the *electrical permittivity* of the material
  - ε_r is the *relative permittivity* of the material
  - "Free space" is what you would get in a vacuum $\varepsilon_r = 1$, so $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}$; this is similar to in air where $\varepsilon_r = 1.0006$
- *Gauss's Law* is one of the postulates: $\vec{\nabla} \cdot \vec{D} = \rho_v$

- In differential form this tells us that at any given point, the divergence of the electric flux density is the same as the volume charge density
- Applying the divergence theorem gives us $\oiint_s \vec{D} \cdot \mathrm{d}\vec{S} = \iiint_V \rho_v \mathrm{d}V = Q_{enc}$
  - In integral form this tells us that the net electric flux through a closed surface is the net charge enclosed by the surface
- Coulomb's Law can be derived from this, if we assume $\vec{D}$ only has a radial component

Lecture 7, Jan 23, 2023

Field Computation Using Gauss's Law

- Using Gauss's law we can solve for the field from the charge distribution, but only if we know the nature of the field beforehand
- We have to make two assumptions:
  1. What components does the field have?
  2. How does the field magnitude change with space? (i.e. Which variables is it a function of?)
- From these questions we can determine the different regions we have to evaluate Gauss's law on and what kind of Gaussian surface is needed
  - A *Gaussian surface* is an imaginary surface on which we find the flux
  - We must choose the Gaussian surface wisely to make questions solvable at all; to do this we need to make use of symmetry
- In order of bring $\vec{D}$ out of the flux integral $\oiint_S \vec{D} \cdot \mathrm{d}\vec{S} = Q_{enc}$, we need:
  1. S has to be closed
  2. S has to be oriented such that $\vec{D} \cdot \mathrm{d}\vec{S} = 0$ or $D\,\mathrm{d}S$
  3. Over points where $\vec{D} \cdot \mathrm{d}\vec{S} = D\,\mathrm{d}S$, $\|\vec{D}\|$ should be constant

Lecture 8, Jan 25, 2023

Electric Scalar Potential

- By bringing repelling charges together or attracting charges apart, we do work that is stored; this is the idea of *electric potential*

Definition

The electric scalar potential, or voltage ΔV between two points is defined as the work done by an external agent per unit charge, or

$$\Delta V = V_2 - V_1 = V_{21} = -\int_{P_1}^{P_2} \vec{E} \cdot \mathrm{d}\vec{l}$$

- In the case where $\vec{E}$ is constant, we just have ΔV being the field strength times distance between the two points
- Note the negative sign: if the electric field does work between the two points, the potential difference is negative; the electric field always points from high potential to low potential
- Consider a point charge Q at the origin and two points P_1 and P_2
  - $\Delta V = -\int_{P_1}^{P_2} \vec{E} \cdot \mathrm{d}\vec{l} = -\int_{P_1}^{P_2} \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R \cdot \mathrm{d}\vec{l}$
  - We can choose our path so that we move radially first, and then move along a sphere; this allows us to get rid of the dot product, because the radial movement is parallel to $\hat{a}_R$ and the spherical movement is perpendicular

- We get $\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$ as the potential difference between two points due to a single point charge
- If we let $R_1 \to \infty$ be our reference, then we just get $\Delta V = V_2 = \frac{Q}{4\pi\varepsilon_0 R_2}$

Definition

The *absolute electric potential* due to a point charge is

$$V(R) = \frac{Q}{4\pi\varepsilon_0 R}$$

This assumes a reference of a charge at $R = \infty$ having zero potential

- Note the expression for the potential is the same as Coulomb's law but the R term is not squared
- A surface which has the same value of V over the entire surface is called an *equipotential surface*
  - This could be a physical surface or an imaginary surface
  - e.g. a sphere surrounding a point charge is an equipotential surface since potential depends only on R; for a dipole these are ellipsoids
  - All perfect conductors are equipotential surfaces
  - The electric field is always perpendicular to equipotential surfaces

Lecture 9, Jan 30, 2023

Electric Scalar Potential of Multiple Charges

- For a point charge not at the origin, we can generalize the expression for electric scalar potential to $V = \frac{Q}{4\pi\varepsilon_0 R} = \frac{Q}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'\|}$
- With multiple point charges, we can take the superposition as $V_{tot} = \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}_i'\|}$
- For a continuous charge distribution we integrate: $V = \int \frac{\mathrm{d}Q'}{4\pi\varepsilon_0 \|\vec{R} - \vec{R}'\|}$

Lecture 10, Feb 1, 2023

Relating the Electric Field and Electric Potential

- Because $\vec{\nabla} \times \vec{E} = 0$ for electrostatic fields (Faraday's Law), it is the gradient of a scalar field
  - This also means that moving in a closed loop does not do any work
- We let $\vec{E} = -\vec{\nabla}\vec{V}$ where $\vec{V}$ is the electric scalar potential
  - The negative sign is to make sure the electric field points away from positive charges
- Note $\vec{E}$ only cares about the spatial rate of change of V, not its actual value, so it doesn't matter if the potential is absolute or relative

Gradient

- The gradient is the direction and magnitude of the maximum spacial rate of change
- The gradient in different coordinate systems is given by:
  - $\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$
  - $\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$

– $\frac{\partial V}{\partial R}\hat{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$

- The factors of $\frac{1}{r}$ etc can be thought of as making sure the gradient has the right units

Lecture 11, Feb 3, 2023

Effect of Electric Field on Materials

- If an external electric field is applied to a material, then "excess" or "mobile" charges will be pushed along by the field
- Based on the amount of mobile charges, most materials fall into 3 categories: conductors, semiconductors, and dielectrics (insulators)
  - In a conductor the band gap is very small so very little energy is needed to promote an electron to the conduction band
  - In a dielectric the band gap is quite large, so a lot of energy is needed for conduction
- The movement of charges creates a current I; we can define a *current density* J so that $I = \iint_S \vec{J} \cdot \mathrm{d}\vec{S}$
  - $\vec{J}$ has units of $\mathrm{A/m^2}$

Equation

The relationship between current density and an electric field causing the current is

$$\vec{J} = \sigma\vec{E}$$

where σ is the conductivity of the material

- This is known as Ohm's law in microscopic (point) form
- Conductivity characterizes how easily a current flows within that material
  - Later we see $\sigma = \frac{N_e e^2 \tau}{m_e}$ where N_e is the electron density and τ is the mean free time
  - In general σ goes down as temperature goes up as τ decreases when the atoms become more energetic
- Resistivity is the inverse of conductivity, $\rho = \frac{1}{\sigma}$ with units of $\Omega\,\mathrm{m}$
- Properties of perfect conductors and dielectrics:
  - In a perfect conductor, $\sigma \to \infty$, so no applied field is needed for current to flow, and *there is always zero electric field*
  - In a perfect insulator, $\sigma \to 0$, so there is never any current; the electric field can be anything but the material will not respond
- A perfect conductor will have the same potential everywhere on its surface, so all perfect conducting surfaces are equipotential; therefore the electric field is always perpendicular to them

Lecture 12, Feb 6, 2023

Polarization

- Consider the effect of a static field on the atoms in an insulator:
  - The field inside the insulator, $\vec{E}_0$, *polarizes* the bond atoms within the material – the atoms are stretched, negative charges and positive charges are pushed to different sides
  - The insulator has become a *dielectric* (refers to the separation of the positive and negative charges)
  - This separation of charge creates an electric field of its own, the *polarization field* $\vec{E}_n$
    - * Now the total field is reduced: $\vec{E}_{tot} = \vec{E}_0 - \vec{E}_p$
  - The polarized atoms can be approximated with an *electric dipole*

- The dipole is represented by the *dipole moment* $\vec{p} = Q\vec{d}$, where $\vec{d}$ is the vector connecting the two charges with $-Q$ and $+Q$
  - $\vec{d}$ and thus $\vec{p}$ always points from the negative charge to the positive charge by convention

Definition

The *dipole moment* of a pair of charges $-Q$ and Q, where $\vec{d}$ is the vector pointing from the negative charge to the positive charge, is

$$\vec{p} = Q\vec{d}$$

The *polarization vector* of a material is

$$\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{N\Delta v} \vec{p}_i \approx N\vec{p}_i \qquad \text{if } \vec{p}_i \text{ are the same}$$

where N is the number of atoms per unit volume

- The polarization vector has units of $\mathrm{C/m^2}$; it is a measure of the "average" dipole moment per unit volume
- In this course we are concerned with *simple media*, that is:
  - Linear: The properties of the dielectric do not depend on the electric field strength
    - The strength of the polarization is directly proportional to the applied field
  - Isotropic: The properties of the dielectric do not depend on the field direction
    - The direction of polarization is always parallel to the direction of the applied field
  - Homogeneous: The properties of the dielectric do not depend on the electric field position (i.e. there's only one type of material)
    - The relationship between the polarization and the electric field is the same everywhere within the material
- For such simple media, $\vec{P} = \varepsilon_0 \chi_e \vec{E}$
  - χ_e is the *electric susceptibility* of the material, a unitless quantity – higher χ_e means higher polarization
    - χ_e can be a matrix if the material is not isotropic
  - Define $\varepsilon_r = \chi_e + 1$ so that $\vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$ where ε_r is the *relative permittivity* of the material
    - Therefore $\vec{D} = \varepsilon\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$
- At the surfaces of the material, there's no more atoms, we have a negative charge layer and a positive charge layer, which causes attraction
  - This results in the *bound charge density* $\pm\rho_{sb}$ (as opposed to the free charge densities $\pm\rho_s$)
  - Therefore $\vec{E}_p = \dfrac{\rho_{sb}}{\varepsilon_0}$ or $\|\vec{E}_{tot}\| = \|\vec{E}_0\| - \|\vec{E}_p\| = \dfrac{1}{\varepsilon_0}(\rho_s - \rho_{sb})$
  - **This is equivalent to reducing it by a factor of ε_r: $\vec{E}_{tot} = \dfrac{\vec{E}_0}{\varepsilon_r}$**
- In a polarized material:
  - $\vec{E}_0$ is the original field applied, which results in free charge densities
  - Bound charge densities $\rho_{sb} = \vec{P} \cdot \vec{a}_n$ where $\vec{a}_n$ are the outward normal vectors of the surface
    - This gives $\rho_{sb} = \|\vec{P}\|$ if the polarization vector is normal to the surface
- When dealing with these problems, it's important to note whether the potential difference or the ρ_s are constant

Summary

To account for polarization in a material, all we need to do is to add the relative permittivity ε_r:

$$\vec{E} = \frac{\vec{E}_0}{\varepsilon_r}$$

The polarization induces bound charges, with surface bound charge density given by

$$\rho_{sb} = \vec{P} \cdot \hat{a}_n$$

where $\hat{a}_n$ is the unit normal of the surface; the volume bound charge density is given by

$$\rho_{vb} = -\vec{\nabla} \cdot \vec{P}$$

$\vec{P}$ is the polarization vector, given by

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E} = \vec{D} - \varepsilon_0 \vec{E}$$

Lecture 13, Feb 8, 2023

Electric Flux Density and Polarization

- $D = \varepsilon_r \varepsilon_0 \vec{E} = \varepsilon_r \varepsilon_0 (\vec{E}_0 - \vec{E}_p), \varepsilon_r = \chi_e + 1, \vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$
- $\chi_e = \dfrac{\rho_{sb}}{\varepsilon_0(E_0 - E_p)} = \dfrac{\rho_{sb}}{\rho_s - \rho_{sb}}$
- $\varepsilon_r = \chi_e + 1 = \dfrac{\rho_s}{\rho_s - \rho_{sb}}$
- $\vec{D} = \left(\dfrac{\rho_s}{\rho_s - \rho_{sb}}\right) \varepsilon_0 \left(\dfrac{\rho_s}{\varepsilon_0} - \dfrac{\rho_{sb}}{\varepsilon_0}\right) = \rho_s$
  - Note this is for a flat plate capacitor
- In the end the electric flux density relates only to the free charge, but the electric field relates to both free and bound charge
- Therefore $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$
  - Note $\vec{D}$ is not something *changed* by polarization; rather it is the total field $\vec{E}$ that changes
  - The combination of changing $\vec{E}$ and polarization $\vec{P}$ produces a constant $\vec{D}$, unaffected by dielectric changes
- $\vec{D}$ is the *electric flux density* or *electric displacement vector*
  - $\vec{D}$ is completely material independent
  - $\vec{D}$ represents the flow, or flux, of the "presence" of charge – it is connected only to the source of the field (i.e. the free charges)
    - This is why $\oiint_S \vec{D} \cdot \mathrm{d}\vec{S} = Q_{enc}$
- $\vec{E}$ is the *electric field intensity*
  - It relates to the total charge, both free and bound
  - $\vec{E}$ comes from the electric force per unit charge
  - $\vec{E}$ represents the effects of the entire field with all of its charges/forces

Lecture 14, Feb 10, 2022

Dielectric Breakdown

- When a strong enough electric field is applied, even in a dielectric the electrons can jump from the valence to the conduction band

- When this happens, the material becomes a conductor; this is referred to as *dielectric breakdown*
- The field is strong enough to overcome the attractive force between the nucleus and its orbiting electrons; the atom goes beyond just stretching and the electrons are detached

- The *dielectric strength* E_b is the maximum electric field that the material can withstand before a current flows
  - E_B for air is 3×10^6 V/m
  - E_B for mica is 200×10^6 V/m
    * This is why mica is used for capacitors – the very small inter-plate distance means the same voltage creates a much larger electric field
- Lightning is a great example of this
  - Lightning rods work by concentrating the electric field at its end

Boundary Conditions for the Electric Field in Materials

- Application example: optical fibres
  - Low conductivity of the glass reduces conductive power loss
  - e.g. copper wire requires signal boosters every 10 km; with optical fibres boosters are only needed every 100 km or 1000 km
  - Optical fibres consists of an outer cylinder (cladding) with an index of refraction on the order of 1.2, and a core cylinder with an index of refraction slightly larger
    * Index of refraction is directly related to ε_r
  - A light source shines into the core, and most of the light is reflected and travels down the core
    * When the light hits the interface between the cladding and the core, total internal reflection happens
  - Total internal reflection happens due to the boundary conditions
  - The fibre is a *waveguide* that carries electromagnetic waves
- Consider the boundary of two materials 1 and 2, to see how an electric field behaves at the boundary we apply Maxwell's equations
- Break the electric field into tangential and normal components; these components are affected differently, and based on how they are affected the field lines bend
- Consider the tangential components $\vec{E}_{t1}, \vec{E}_{t2}$
  - From Faraday's law $\oint_C \vec{E} \cdot \mathrm{d}\vec{l} = 0$
  - We can create a contour right on the boundary with infinitesimal thickness, so we can isolate the boundary tangential components
  - $\oint_C \vec{E} \cdot \mathrm{d}\vec{l} = E_{t2}\Delta l - E_{t1}\Delta l = 0 \implies E_{t1} = E_{t2} \implies \dfrac{D_{t1}}{\varepsilon_{r1}} = \dfrac{D_{t2}}{\varepsilon_{r2}}$
  - At the boundary of an interface, tangential components of the field have to be the same
- Consider the normal components $\vec{D}_{n1}, \vec{D}_{n2}$
  - From Gauss's law $\oiint_S \vec{D} \cdot \mathrm{d}\vec{S} = Q_{enc}$
  - Use a Gaussian cylinder, with parts just above and just below the boundary, with surface area ΔS
  - Through this cylinder we have D_{n2} going in from the bottom and D_{n1} coming out from the top
  - In the limit: $\oiint_S \vec{D} \cdot \mathrm{d}\vec{S} = -D_{n2}\Delta S + D_{n1}\Delta s = \rho_s \Delta S \implies D_{n1} - D_{n2} = \rho_s$ or $\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$
  - This gives us $\varepsilon_{r1}\varepsilon_0 E_{n1} - \varepsilon_{r2}\varepsilon_0 E_{n2} = \rho_s$
  - When there is no surface free charge, $\varepsilon_{r1}E_{n1} = \varepsilon_{r2}E_{n2}$

Summary

At the boundary between two dielectrics, taking the normal direction to be pointing from material 2 to material 1:

$$E_{t1} = E_{t2} \implies \frac{D_{t1}}{\varepsilon_{r1}} = \frac{D_{t2}}{\varepsilon_{r2}}$$

$$D_{n1} - D_{n2} = \rho_s \implies \varepsilon_{r1}\varepsilon_0 E_{n1} - \varepsilon_{r2}\varepsilon_0 E_{n2} = \rho_s$$

or when there is no free charge at the boundary:

$$D_{n1} = D_{n2} \implies \varepsilon_{r1} E_{n1} = \varepsilon_{r2} E_{n2}$$

Lecture 15, Feb 13, 2023

Lecture 16, Feb 15, 2023

Capacitance

Definition

Given two conductors with a potential difference, the capacitance C between them is defined as

$$C = \frac{Q}{\Delta V} = \frac{Q}{V}$$

with units of $[\mathrm{C/V} = \mathrm{F}]$

- Consider 2 conductors attached to a battery; they will have equal and opposite charges proportional to the voltage
- Let $Q = C\Delta V$; C is the *capacitance*, $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{V}[\mathrm{C/V} = \mathrm{F}]$
  - Capacitance is a function of only the conductor geometry and the material separating them
  - A large capacitance results in large Q for a small V
  - Application notes: typically in circuits there are lots of conductors next to each other, which can introduce a parasitic capacitance; this capacitance can severely distort high frequency signals
- $C = \dfrac{Q}{\Delta V} = \dfrac{\oiint_{S+} \vec{D} \cdot \mathrm{d}\vec{S}}{\left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right|} = \dfrac{\oiint_{S+} \varepsilon_r\varepsilon_0 \vec{E} \cdot \mathrm{d}\vec{S}}{\left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right|}$
  - $S+$ is the surface that encloses the positively charged conductor but this could be another conductor as well
- Example: parallel plate capacitor filled with dielectric with relative permittivity ε_r
  - First we need to find the electric field in the dielectric: $\vec{E} = \dfrac{\rho_s}{\varepsilon_0\varepsilon_r}$
    * We can see this from boundary conditions $D_{\text{dielectric}} - D_{\text{conductor}} = \rho_s \implies D = D_{\text{dielectric}} = \rho_s$
  - $\Delta V = \left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right| = Ed = \dfrac{\rho_s d}{\varepsilon_0\varepsilon_r} = \dfrac{Qd}{\varepsilon_0\varepsilon_r S}$
    * $Q = \rho_s S$ where S is the area of the plate, assuming uniform free charge distribution
  - Therefore $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{\frac{Qd}{\varepsilon_0\varepsilon_r S}} = \dfrac{\varepsilon_0\varepsilon_r S}{d}$
  - This assumes uniform charge distribution (i.e. plates are effectively infinitely large), and also the insulation cannot conduct any current
- To maximize capacitance we increase ε_r or the plate area S, and making the plate spacing D as small as possible
- Example: capacitance of a spherical capacitor; inner sphere with radius a, outer sphere with radius b,

dielectric with ε_r between

  - Assume a Q, find $\vec{E}$ from it, and then find ΔV
  - Assume inner shell has charge $+Q$ and outer shell has charge $-Q$
  - Apply Gauss's law: $\oiint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q_{enc}}{\varepsilon_0 \varepsilon_r} \implies 4\pi\varepsilon_r\varepsilon_0 R^2 E_R = Q \implies \vec{E} = \frac{Q}{4\pi R^2 \varepsilon_r \varepsilon_0}\hat{a}_R$
  - Find potential by integration: $\Delta V = \left| \int_a^b E \,\mathrm{d}R \right| = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$
  - Therefore $C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_r\varepsilon_0 ab}{b-a}$
  - Does this make sense?
    * C is directly proportional to ε_r
    * The distance between conductors in the denominator
    * The surface area of the conductors is in the numerator

Summary

To find capacitance:

1. Assume some charge Q on the surfaces
2. Use boundary conditions and Gauss's Law to find $\vec{D}$ and $\vec{E}$
3. Find voltage $\Delta V = \left| \int \vec{E} \cdot \mathrm{d}\vec{l} \right|$
4. Calculate capacitance by $C = \frac{Q}{\Delta V}$

Lecture 17, Feb 17, 2023

Electrostatic Energy

- The work done to bring a charge from infinity in is $-\int_{-\infty}^{P_2} \vec{F}_{12} \cdot \mathrm{d}\vec{l} = -Q_1 \int_{-\infty}^{P_2} \vec{E} \cdot \mathrm{d}\vec{l} = Q_1 V(P_2)$
  - If we being in another charge we have to account for the repulsion of the additional charges already there

Definition

For a collection of point charges, the total stored energy is

$$W_e = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$

For a continuous charge distribution this is

$$W_e = \frac{1}{2}\iiint_V \rho_v V \,\mathrm{d}V$$

where V is the potential of the total system after all the charges have been brought together

In terms of the fields, from $\vec{\nabla}\cdot\vec{D} = \rho_v$ and $\vec{E} = -\vec{\nabla}V$, we have

$$W_e = \frac{1}{2}\iiint_V \vec{D}\cdot\vec{E}\mathrm{d}\vec{V} = \frac{1}{2}\iiint_V \varepsilon|\vec{E}|^2\,\mathrm{d}V$$

Where the energy density is

$$W_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon_r\varepsilon_0|\vec{E}|^2 = \frac{1}{2}\frac{|D|^2}{\varepsilon_r\varepsilon_0}$$

- The factor of $\frac{1}{2}$ accounts for duplication between charge interactions
- Consider the energy stored in a parallel plate capacitor:
  - First method: using charges
    - $W_e = \frac{1}{2}\iint \rho_s V\,\mathrm{d}S = \frac{1}{2}\iint_S \rho_s V_0\,\mathrm{d}S = \frac{1}{2}\rho_s V_0 S = \frac{1}{2}QV_0 = \frac{1}{2}CV_0^2 = \frac{1}{2}\frac{\varepsilon_r\varepsilon_0 S}{d}V_0^2$
  - Second method: using fields
    - $W_e = \frac{1}{2}\iiint_V \varepsilon_r\varepsilon_0|\vec{E}|^2\,\mathrm{d}V$
    - For a parallel plate capacitor $\vec{E}$ has constant magnitude $\frac{\rho_s}{\varepsilon_r\varepsilon_0}$, and the volume is Sd
    - Therefore $W_e = \frac{1}{2}\left(\frac{\rho_s}{\varepsilon_0\varepsilon_r}\right)^2\varepsilon_r\varepsilon_0 Sd = \frac{1}{2}\frac{Q^2}{S\varepsilon_r\varepsilon_0}Sd = \frac{1}{2}\frac{Q^2 d}{\varepsilon_r\varepsilon_0 S} = \frac{1}{2}\frac{Q^2}{C}$

Important

$W_e = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV_0^2$ holds in general; we may find C from energy by $C = \frac{1}{2}\frac{Q^2}{W_e} = \frac{2W_e}{V_0^2}$

Lecture 18, Feb 27, 2023

Laplace's and Possion's Equations

- Since $\vec{\nabla}\cdot\vec{D} = \rho_v$, we have $\vec{\nabla}\cdot(\varepsilon\vec{E}) = \rho_v$
- $\vec{\nabla}\times\vec{E} = 0 \implies \vec{E} = -\vec{\nabla}V$
- Combining these we get Possion's equation: $\vec{\nabla}\cdot(\varepsilon\vec{\nabla}V) = -\rho_v$
- In the case where $\rho_v = 0$ we get Laplace's equation: $\vec{\nabla}\cdot(\varepsilon\vec{\nabla}V) = 0$
- If the electric field is homogeneous, then we may take out ε and get $\vec{\nabla}\cdot\vec{\nabla}V = -\frac{\rho_v}{\varepsilon} \implies \vec{\nabla}^2 V = -\frac{\rho_v}{\varepsilon}$

Equation

Possion's equation:

$$\vec{\nabla} \cdot (\varepsilon \vec{\nabla} V) = -\rho_v$$

In a homogeneous material, this reduces to

$$\vec{\nabla}^2 V = -\frac{\rho_v}{\varepsilon}$$

Where there is no charge density, these are called Laplace's equations

Lecture 19, Mar 1, 2023

Boundary Value Problems

- Motivation: usually we don't have any idea what the charge distribution ρ_s is like
- We often know what values of V are on the boundaries of the problem
- By using Laplace's or Poisson's equations we can determine $\vec{E}$ in a given problem without knowing the charge densities
- Example: parallel plate capacitor
  - Assume $\rho_v = 0$ and ε_r is constant, so we use Laplace's equation $\vec{\nabla}^2 V = 0$
  - Assume $\vec{E} = E\hat{a}_x$, then $\vec{\nabla}^2 \vec{V} = \dfrac{\mathrm{d}^2 V}{\mathrm{d}x^2} = 0 \implies V(x) = c_1 x + c_2$
  - Using boundary conditions $V(0) = V_0, V(d) = 0$ we get $V(x) = -\dfrac{V_0}{d}x + V_0$
- In general, start with Poisson's equation; if the field is homogeneous we can take out ε; if there is no charge density then we can use Laplace's equation
- Then use the equation to double integrate to find V, using boundary conditions to find the constants, then find $\vec{E}$
- Finally from E we may find other quantities such as Q with a variety of methods (Gauss's Law, boundary conditions, finding work, etc)

Lecture 20, Mar 3, 2023

Uniqueness Principle

Theorem

If a solution to Laplace's or Poisson's equation can be found that satisfies the boundary conditions, then the solution must be unique

- Example: electric shielding
  - What is the voltage and electric field inside a closed conductive shell with no enclosed charge?
  - We need Laplace's equation, with a homogeneous medium $\vec{\nabla}^2 V = 0$
  - Boundary condition: $V = V_0$ everywhere on the boundary of the shell since it's a perfect conductor
  - The simplest solution is $V(x, y, z) = V_0$; then by the uniqueness principle, we may conclude that this is the only solution
  - Therefore $\vec{E} = -\vec{\nabla} V = 0$, i.e. there is no electric field inside the shell at all
  - More commonly known as a *Faraday cage*
    * This would still work even in a cage where there are holes in the conductor; as long as the holes are smaller than the wavelength of the signal, EM waves will be completely blocked out
    * In practice the thickness of the shell also matters

Method of Images

- Before computers, this method was used to solve challenging EM problems
- When a charge distribution is placed near a perfectly conducting object, the distribution gets "reflected" in the object, i.e. the field configuration resembles that of a dipole
  - There isn't actually any field in the perfect conductor, but the field outside the conductor is identical to the case where we have a dipole
  - This is caused by the fact that the perfect conductor is an equipotential surface so the field lines are perpendicular to it, just line the field lines would be all in the same direction halfway between a dipole
- The charge distribution is reflected and the charge is inverted
- We can take this further and consider the case where the plate is finite, where we have a sphere, where we have multiple conductors, etc

Lecture 21, Mar 6, 2023

Electric Field Inside Conducting Materials

- When an electric field is applied to a material where there are free charge carriers, it will create a current density $\vec{J}$
  - Electrons move with a drift velocity $\vec{u}_d$
- The force on each conductor is $\vec{F}_e = -e\vec{E} = m_e\vec{a} \implies \vec{a} = -\frac{e}{m_e}\vec{E}$
- The charge density is $\rho_{ve} = -N_e e$ where N_e is the charge carrier density; so define the current density as $\vec{J} = \rho_{ve}\vec{u}_d$
  - Current density has units of $\mathrm{A/m^2}$ which this relation satisfies
- By convention current density $\vec{J}$ is in the same direction as the electric field $\vec{E}$
  - $\vec{u}_d$ would be in the opposite direction as $\vec{E}$ and $\vec{J}$ if the charge carriers were electrons
- How can we model the movement of electrons?
  - Consider the case where there is no $\vec{E}$ field applied, electrons move by thermal agitation and bounce around atoms
    * There is no net movement since the movements are completely random
    * The velocity is on the order of 1×10^5 m/s but there is no coordination in direction, so no net movement
  - When a field is applied, there is a net movement in the direction that the field pushes the electrons in
    * The overall average velocity is the drift velocity $\vec{u}_d = \Delta t\vec{a} \approx \tau\vec{a} = -\frac{\tau e\vec{E}}{m_e}$
    * τ is the mean free time, or average time between collisions
  - Since $\vec{u}_d$ is directly connected to current density, higher τ means better conductor
  - Define the mobility $\mu_e = -\frac{e\tau}{m_e}$ so that $\vec{u}_d = \mu_e\vec{E}$
    * The mobility takes into account both τ and the type of charge carrier
- Since the current density is current per unit area, $I = \iint_S \vec{J} \cdot \mathrm{d}\vec{S}$
- Therefore $\vec{J} = \rho_{ve}\vec{u} = -N_e e\left(-\frac{\tau e}{me}\right) = \frac{N_e e^2\tau}{m_e}\vec{E} = \sigma\vec{E}$

Definition

Ohm's Law in point form:

$$\vec{J} = \sigma\vec{E}$$

where $\sigma = \frac{N_e e^2\tau}{m_e}$ is the conductivity of the material

- Using this we can derive another equation for the boundary condition: $E_{t1} = E_{t2} \implies \frac{J_{t1}}{\sigma_1} = \frac{J_{t2}}{\sigma_2}$
  - Combine this to with the boundary condition for $\vec{D}$ we have $\frac{\varepsilon_{r1}\varepsilon_0 J_{n1}}{\sigma_1} - \frac{\varepsilon_{r2}\varepsilon_0 J_{n2}}{\sigma_2} = \rho_s$
  - For a steady current interface, J is continuous: $J_{n1} = J_{n2} = J_n$, therefore $\rho_s = J_n \left(\frac{\varepsilon_{r1}\varepsilon_0}{\sigma_1} - \frac{\varepsilon_{r2}\varepsilon_0}{\sigma_2}\right)$

Summary

Electric current quantities:
- N_e charge carrier density (number density of charge carriers)
- $\rho_{ve} = -N_e e$ charge density (density of moving charges)
- $\vec{J} = \rho_{ve}\vec{u}_d$ current density (current per unit area)
- $\vec{u}_d = \mu_e \vec{E} = -\frac{\tau e \vec{E}}{m_e}$ drift velocity (average velocity of moving electrons)
- $\mu_e = -\frac{e\tau}{m_e}$ (electron) mobility (how easily electrons move given an applied electric field)
- $\sigma = \frac{N_e e^2 \tau}{m_e}$ conductivity

Summary

Boundary conditions for current density for a current going from material 2 to material 1:
- Tangential component: $\frac{J_{t1}}{\sigma_1} = \frac{J_{t2}}{\sigma_2}$
- Normal component: $\frac{\varepsilon_{r1}\varepsilon_0 J_{n1}}{\sigma_1} - \frac{\varepsilon_{r2}\varepsilon_0 J_{n2}}{\sigma_2} = \rho_s$
- Given a steady current interface, we can find $\rho_s = J_n \left(\frac{\varepsilon_{r1}\varepsilon_0}{\sigma_1} - \frac{\varepsilon_{r2}\varepsilon_0}{\sigma_2}\right)$

Lecture 22, Mar 8, 2023

Resistance and Conductance

- Recall that $\vec{J} = \sigma\vec{E}, \vec{E} = \rho\vec{J}$ where σ is the conductivity and ρ is the resistivity, with $\frac{1}{\rho} = \sigma$
- Resistance and conductance are macroscopic properties that apply to an entire piece of material rather than points within the material
- Using Ohm's law we define resistance as $R = \frac{V}{I}$, conductance as $G = \frac{I}{V} = \frac{1}{R}$
- Consider a material with conductivity σ connected to a battery with voltage V; this creates a field $\vec{E}$ that generates a current $\vec{J}$
  - We know $V = -\int \vec{E} \cdot \mathrm{d}\vec{l}$ and $I = \iint_S \vec{J} \cdot \mathrm{d}\vec{s} = \iint_S \sigma\vec{E} \cdot \mathrm{d}\vec{s}$ where S is the cross-sectional area
  - Therefore $R = \frac{V}{I} = \frac{\left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right|}{\iint_S \sigma\vec{E} \cdot \mathrm{d}\vec{s}}$
- Like capacitance, to find the resistance of any material we can assume some voltage and compute how much current this creates, and then take the ratio
- In the case of a simple conductor with uniform S and σ, $V = EL, I = E\sigma S \implies R = \frac{V}{I} = \frac{L}{\sigma S}$
  - For the uniform area conductor we know $\vec{E}$ is constant, because I is constant and therefore $\vec{J}$ must be constant, which leads to $\vec{E}$ being constant
  - This applies to e.g. a cylinder, but not a cone

  - If we don't have uniform cross-sectional area, $R = \int \frac{1}{\sigma S}\,\mathrm{d}l$ where S and σ can be functions of l
    * This works even in the case of a non-uniform electric field
    * Can be thought of as a collection of infinitesimal resistors in series
- Example: resistance of a coaxial cable filled with dielectric ε_r
  - Use Gauss's law to find $\vec{E} = \frac{\rho_{sa} a}{\varepsilon_r \varepsilon_0 r}\hat{a}_r$
  - $R = \frac{\left|\int \vec{E}\cdot \mathrm{d}\vec{l}\right|}{\iint_S \sigma \vec{E}\cdot \mathrm{d}\vec{s}} = \frac{\left|\int \frac{\rho_{sa} a}{\varepsilon_r\varepsilon_0 r}\,\mathrm{d}r\right|}{\int_0^{2\pi}\int_0^L \frac{\sigma \rho_{sa} a}{\varepsilon_r \varepsilon_0 a} a\,\mathrm{d}z\,\mathrm{d}\phi} = \frac{\ln\frac{b}{a}}{2\pi L\sigma}$
  - When we're evaluating the bottom integral to find current, we can choose any surface; usually we choose one so that $\vec{J}\cdot\mathrm{d}\vec{s}$ is easy to evaluate (e.g. a Gaussian surface)
    * We only need to integrate over a single surface, in this case a cylinder of radius a
  - We can also find this by $R = \int \frac{1}{\sigma S}\,\mathrm{d}l = \int \frac{1}{\sigma \cdot 2\pi r L}\,\mathrm{d}r$
  - Note this assumes that the electric field is uniform down the wire; if the wire were extremely long we would need to consider how $\vec{E}$ changes as you move down the wire, due to resistance of the conductor and leakage current through the dielectric

Joule's Law (Power Loss)

- Since current is the result of the electric field doing work on the electrons, the electric field has to do work to create current
- Power is lost as heat in the system
- Consider an electric field causing a current $\vec{J} = \sigma\vec{E}$ in a non-perfect dielectric; how much power does it take to sustain this current?
  - Consider a very small region; the charges move at the drift velocity
  - $\Delta P = \frac{\mathrm{d}\Delta U}{\mathrm{d}t}$ and $\Delta U = W_e$ so $\Delta P = \frac{\mathrm{d}}{\mathrm{d}t}\int \vec{F}_e\cdot\mathrm{d}\vec{l} = \frac{\mathrm{d}}{\mathrm{d}t}\int Q\vec{E}\cdot\mathrm{d}\vec{l} = \frac{\mathrm{d}}{\mathrm{d}t}\int \rho_v \Delta v\vec{E}\cdot\mathrm{d}\vec{l}$
  - $\Delta P = \int \rho_v \Delta v \vec{E}\cdot\frac{\mathrm{d}\vec{l}}{\mathrm{d}t} = \vec{E}\cdot\vec{J}\Delta v$

Definition

Joule's law:

$$P = \iiint_V \vec{E}\cdot\vec{J}\,\mathrm{d}V$$

relates energy losses in a conductor to the current and electric field in it

Lecture 23, Mar 13, 2023

Magnetostatics

- The magnetic field intensity $\vec{B}$ is created by a current I according to the right hand rule; the field forms a loop around the current-carrying conductor}
- A fundamental postulate is $\vec{\nabla}\cdot B = 0$; this means that the magnetic field intensity vector always forms closed loops
- By convention, $\vec{B}$ emanates from the *north* pole and ends at the *south* pole

Definition

The Lorentz force law: The force felt by a charge q moving with velocity $\vec{u}$ in a magnetic field $\vec{B}$ is

$$\vec{F}_m = q\vec{u} \times \vec{B}$$

This makes the total force on a charge moving in an electric and magnetic field

$$\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B}$$

- Note that a charge that is not moving does not feel any magnetic force; also since the magnetic force is normal to the direction of velocity, the magnetic force cannot change the speed, only the direction of a moving charge
- For a current carrying wire, $q\vec{u} = \int I \, \mathrm{d}\vec{l} = I\vec{L}$ so $\vec{F}_m = I\vec{L} \times \vec{B}$
  - Therefore two parallel wires will attract each other if they carry currents in the same direction, and repel each other if they carry currents in different directions

Lecture 24, Mar 15, 2023

Magnetic Vector Potential

- We know $\vec{\nabla} \cdot \vec{B} = 0 \implies \oiint_S \vec{B} \cdot \mathrm{d}\vec{s} = 0$
  - Magnetic flux through a closed surface is always zero
  - Magnetic flux is denoted by Φ_m
- Since the divergence of curl is 0 we can express $\vec{B}$ as the curl of a potential $\vec{A}$
  - $\vec{B}$ has units of tesla, or webers per unit area (where weber is the unit of magnetic flux, in volt seconds)
  - $\vec{A}$ has units of tesla meters, or webers per unit length
  - Note $\vec{A}$ does not have relation to energy like V does

Definition

The magnetic vector potential $\vec{A}$ is defined such that

$$\vec{B} = \nabla \times \vec{A}$$

It is directly related to the current as

$$\vec{\nabla}^2\vec{A} = -\mu_0\vec{J} \implies \vec{A} = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}}{|\vec{R} - \vec{R}'|}\,\mathrm{d}v' = \frac{\mu_0}{4\pi}\int_L \frac{I}{|\vec{R} - \vec{R}'|}\,\mathrm{d}l'$$

which is the analogue of the Poisson equation, where μ_0 is the magnetic permeability

- Note that the magnetic vector potential is always in the same direction as $\vec{J}$
- The magnetic flux can be directly determined from the magnetic vector potential:
  - $\Phi_m = \oint_C \vec{A} \cdot \mathrm{d}\vec{l}$ where Φ_m is the magnetic flux through any surface with C as its boundary
  - This follows directly from Stokes' theorem

The Biot-Savart Law

- All magnetic phenomenon come from moving charges (in a permanent magnet, this comes from the movement of charges in the atoms)

- The Biot-Savart law relates magnetic fields to their sources
- Consider a very small bit of current (a current element or filament, which is part of a larger current loop)
    - This bit of current has position $\vec{R}'$ and creates a field at $\vec{R}$

Definition

The Biot-Savart law relates the magnetic field intensity to currents:

$$\mathrm{d}\vec{B} = \frac{\mu_0 I \,\mathrm{d}\vec{l} \times (\vec{R} - \vec{R}')}{4\pi|\vec{R} - \vec{R}'|^3}$$

- $I\,\mathrm{d}\vec{l}$ is the analogue of $\mathrm{d}Q$
- This is completely analogous to Coulomb's law, except for the cross product, which represents the right hand rule
- Given different types of current distributions we can integrate this in different ways to find $\vec{B}$:
    - Moving charge: $\vec{B} = \frac{\mu_0}{4\pi}\frac{Q\vec{u} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$
    - Linear current loop: $\vec{B} = \frac{\mu_0}{4\pi}\int_C \frac{I\,\mathrm{d}\vec{l}' \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$
    - Surface current: $\vec{B} = \frac{\mu_0}{4\pi}\iint_S \frac{\vec{J}_s \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}\,\mathrm{d}s'$
    - Volume current: $\vec{B} = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}\mathrm{d}v'$
- The Biot-Savart law can be derived from $\vec{A}$
- e.g. for a strip of length $2a$, in the x-y plane extending infinitely in the x direction, the current is $\vec{J}_s\,\mathrm{d}s' = \frac{I}{2a}\,\mathrm{d}x'\,\mathrm{d}y'\,\hat{a}_x$; find the field at $P(0, 0, z)$
    - $\mathrm{d}\vec{B} = \frac{\mu_0 \vec{J}_s \times (\vec{R} - \vec{R}')\,\mathrm{d}s'}{|\vec{R} - \vec{R}'|^3}$
    - Integrate in x', y' since those are the dimensions the strip lives in, $\mathrm{d}s' = \mathrm{d}x'\,\mathrm{d}y'$
    - $\vec{J}_s\,\mathrm{d}s' = \left(\frac{I}{2a}\vec{a}_x\right)\mathrm{d}x\,\mathrm{d}y$
    - $\vec{R} = z\vec{a}_z, \vec{R}' = x'\vec{a}_x + y'\hat{a}_y$
    - $\vec{B} = \int_{-\infty}^{\infty}\int_{-a}^{a} \frac{\mu_0\left(\frac{I}{2a}\right)\hat{a}_x \times (-x'\hat{a}_x - y'\hat{a}_y + z\hat{a}_z)}{4\pi\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}}\,\mathrm{d}y'\,\mathrm{d}x'$
    $= \frac{\mu_0 I}{8\pi a}\int_{-\infty}^{\infty}\int_{-a}^{a}\frac{-y'\hat{a}_z - z\hat{a}_y}{\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}}\,\mathrm{d}y'\,\mathrm{d}x'$
    $= -\frac{\mu_0 I}{2\pi a}\tan^{-1}\left(\frac{a}{z}\right)\hat{a}_y$
        * Note we could ignore the $\hat{a}_z$ component because from symmetry and right hand rule we know the field is going to be in the $-\hat{a}_y$ direction

Lecture 25, Mar 17, 2023

Ampere's Law

Definition

Ampere's Law in differential form is given by:

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \left(\frac{1}{\mu_r \mu_0} \vec{B}\right) = \vec{J}$$

Where the magnetic field intensity $\vec{H}$ is related to the magnetic flux density $\vec{B}$ as

$$\vec{B} = \mu_r \mu_0 \vec{H}$$

In integral form, this is

$$\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = \iint_S \vec{J} \cdot \mathrm{d}\vec{s} = I_{enc}$$

- Ampere's law is a fundamental law, the analogue of Gauss's law
- At every point in space, the magnetic field intensity has a nonzero curl only if a current density $\vec{J}$ is present at that point
- The integral form tells us that if we take any contour integral of $\hat{H}$, it is equal to the current crossing through the surface enclosed by that curve
  - Note that the direction of $\mathrm{d}\vec{s}$ in relation to C is defined based on the right hand rule (coming from Stokes' theorem), which is what gives us the right hand rule for $\vec{B}$
- To find $\vec{H}$ from $\vec{J}$ is like finding $\vec{E}$ using Gauss's law; instead of using a Gaussian surface, we use an Amperian loop
  - Choose the loop so that:
    * $\vec{H}$ is always tangential or normal to the loop
    * $\vec{H}$ has a constant value where $\vec{H}$ is a tangential
  - This means $\displaystyle\int \vec{H} \cdot \mathrm{d}\vec{l} = \int H \,\mathrm{d}l = HL$ where L is the length of the loop where $\hat{H}$ is tangential to the loop
- For an infinitely long wire in the $\vec{a}_z$ direction, we choose the Amperian loop to be a circle centered on the wire, which gets us $\displaystyle\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = 2\pi r H_\phi = I_{enc} \implies \vec{H} = \frac{I_0}{2\pi r}\hat{a}_\phi$

Lecture 26, Mar 20, 2023

Magnetic Dipole

- A *magnetic dipole* is simply a closed loop of current, characterized by its *magnetic dipole moment* $\vec{m} = IS\hat{a}_n$, where the direction is determined through the right hand rule and S is the enclosed area
  - e.g. for a loop with radius a, we have $\vec{m} = \pi a^2 I \hat{a}_z$
  - If the loop has n turns, then the effective I is increased, so the magnetic dipole moment is magnified by a factor of n
  - $\vec{m} = nIS\hat{a}_n$ with units $[\mathrm{A\,m^2}]$
  - A magnetic dipole will produce a field in the same direction as the direction it points in
- What happens to a magnetic dipole moment in a $\vec{B}$ field?
  - The loop will experience some magnetic force $\vec{F} = I\vec{L} \times \vec{B}$ (from $\vec{F}_m = q\vec{u} \times \vec{B}$)
  - This produces a net torque $\vec{T} = \vec{m} \times \vec{B}$
  - When $\vec{m}$ and $\vec{B}$ are aligned, the torque goes to zero; therefore a magnetic dipole will rotate until its own field is aligned with the applied field

Magnetization

- All materials have small atomic magnetic dipoles caused by the movement of electrons around the nuclei
  - Since they're all randomly oriented, there is no net field
- In a magnetic material, in the presence of an external magnetic field, the dipoles experience a torque that aligns them in the same direction as the field
- The overall result is that the small $\vec{B}$ fields from the dipoles now all point in the same direction, producing a net magnetic field
  - The magnetic field produced by the dipoles is in the same direction as the external applied field, so they add together
- A material is *magnetic* if it allows their atomic magnetic dipoles to be all aligned in the same direction
- Define the *magnetization vector* $\vec{M}$ (akin to $\vec{P}$'s relationship with $\vec{p}$) as an average of the magnetic dipoles within a material:
  - $\vec{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_i \vec{m}_i \approx N\vec{m}$ with units [A/m]
- This magnetization leads to a *bound current density* (surface) $\vec{J}_{ms} = \vec{M} \times \hat{a}_n$ where $\hat{a}_n$ is the outward normal of the surface
  - There could also be volume bound current densities
- Now we can define 3 new quantities:
  - The magnetic field intensity $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu_r \mu_0}$
  - The magnetic susceptibility χ_m, where $\vec{M} = \chi_m \vec{H}$
  - The relative permeability $\mu_r = \chi_m + 1$
  - Like in the electric field case, $\vec{B}$ accounts for both bound and free currents, but $\vec{H}$ only cares about free currents

Important

The magnetic flux density in a magnetized material is not always greater than the applied field, since the magnetic dipole moments can also align to be antiparallel to the applied field, depending on the material (as a consequence, χ_m is not necessarily positive, so μ_r could be less than 1)

- Example: cylindrical permanent magnet, where a constant uniform $M = M_0\hat{a}_z$ exists; the cylinder is defined by $-\frac{L}{2} \le z \le \frac{L}{2}, 0 \le r \le a$
  - $\vec{J}_{ms} = \vec{M} \times \hat{a}_n = \vec{M} \times \hat{a}_r = M_0\hat{a}_\phi$
  - $$\begin{aligned} \vec{B} &= \iint \frac{\mu_0 \vec{J}_{ms} \times (\vec{R} - \vec{R}')}{4\pi|\vec{R} - \vec{R}'|^3} a \, \mathrm{d}\phi' \, \mathrm{d}z' \\ &= \frac{\mu_0}{4\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^{2\pi} \frac{M_0\hat{a}_\phi \times (-a\hat{a}_r + (z - z')\hat{a}_z)}{(a^2 + (z - z^2)^2)^{\frac{3}{2}}} a \, \mathrm{d}\phi' \mathrm{d}z' \\ &= \frac{\mu_0 M_0}{2} \left(\frac{\frac{L}{2}}{\sqrt{a^2 + (z - L/2)^2}} + \frac{\frac{L}{2}}{\sqrt{a^2 + (z + L/2)^2}} \right) \hat{a}_z \end{aligned}$$
  - What if $L \gg a$?
    * $\vec{B} \to \mu_0 M_0 \hat{a}_z$

Summary

When a magnetic material is exposed to an external applied magnetic field, it is magnetized; the magnetization is characterized by the magnetization vector,

$$\vec{M} = \chi_m \vec{H}$$

where the magnetic field intensity is defined as

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu_r \mu_0}$$

where the relative permeability is defined as

$$\mu_r = \chi_m + 1$$

The magnetization creates a surface bound current density,

$$\vec{J}_{ms} = \vec{M} \times \hat{a}_n$$

where $\hat{a}_n$ is the surface normal vector, and also a volume bound current density,

$$\vec{J}_m = \vec{\nabla} \times \vec{M}$$

Lecture 27, Mar 22, 2023

Generalized Ampere's Law

- Ampere's law becomes $\oint_C \vec{B} \cdot \mathrm{d}\vec{l} = \mu_r \mu_0 I_{enc}$
  - Once again the $\vec{B}$ field is affected by the presence of the material μ_r, but $\vec{H}$ is not
- Example: field inside a solenoid
  - Consider a very long solenoid with n turns per meter filled with a magnetic material with relative permeability of μ_r, with current I_0 through the wire; what is $\vec{H}, \vec{B}$ inside the solenoid?
  - $\vec{H}, \vec{B}$ will be in the same direction, based on RHR, let this be $\hat{a}_z$ so $\vec{B} = B_z \hat{a}_z$
  - Using Ampere's law, with a contour along the edge of the solenoid of length w that encloses the wire
  - When the solenoid is infinitely long, there is no magnetic field outside
  - Therefore $\oint_C \vec{B} \cdot \mathrm{d}\vec{l} = wB_z$ since only the piece of the contour inside the material gives a nonzero dot product
  - The enclosed current is $I_0 nw$
    * For n turns per meter, width of w, nw is the number of turns; therefore nwI_0 is the total current for all these loops
  - $wB_z = \mu_r \mu_0 nwI_0 \implies B = \mu_r \mu_0 n I_0 \hat{a}_z$
  - If we have N turns over L meters, then $\vec{B} = \frac{\mu_r \mu_0 I_0 N}{L} \hat{a}_z$
  - $\vec{H} = nI_0 \hat{a}_z = \frac{NI_0}{L} \hat{a}_z$

Ferromagnetic Materials

- On an atomic level, there are 2 major sources of magnetic dipoles:
  - Orbital motion of the electrons around the nucleus
    * This gives an orbital magnetic dipole moment m_0

- Electron spin
    - This gives a spin magnetic dipole moment m_s
    - Thee two states of spin means that m_s is either parallel or antiparallel to the applied field
- Materials with non-zero internal moments can align ($\mu_r > 1$)
  - *Ferromagnetic* materials have their fields greatly enhanced (strong alignment) ($\mu_r \gg 1$)
  - *Paramagnetic* materials have their fields only slightly enhanced (weak alignment) $\mu_r \approx 1, \mu_r > 1$
  - *Ferrimagnetic* materials are in-between and have $\mu_r > 1$ but not too big; they're useful for higher frequency circuits (e.g. ferrites)
- Materials with zero internal moments actually reduces the net magnetic field ($\mu_r < 1$)
  - *Diamagnetic materials* will have a field in the opposite direction and get repelled by the applied field ($\mu_r \approx 1, \mu_r < 1$)
    - In superconducting materials there will be perfect diamagnetism (the field is perfectly canceled inside the material); this causes levitation (Meissner effect)

Hysteresis

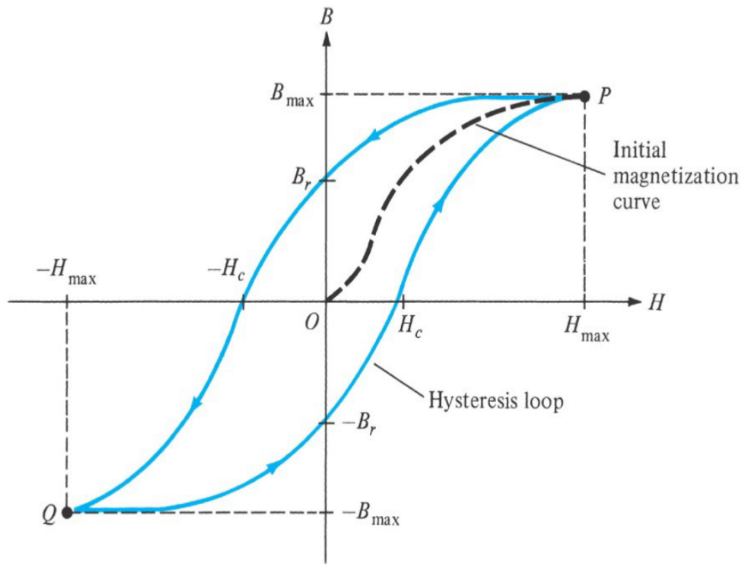

Figure 1: Hysteresis

- When a ferromagnetic material is magnetized, eventually it saturates and B begins to level off even with increasing H
- When the external field is turned off, B goes back down to B_r, the residual flux density – even though there's no more external field, the material stays magnetized
- At this point if we reverse the external current, we first reach the coercive H field or H_c, where the magnetization field disappears
  - At this point the permanent magnetization disappears
- If our applied field $\vec{H}$ varies with time (e.g. a sinusoidal AC current), we will go through the cycle of magnetization-demagnetization over and over
  - This leads to significant energy losses, which we can show to be equal to the area of the hysteresis curve
- *Soft* magnetic materials have smaller B_r values and narrower hysteresis curves, while *hard* magnetic materials have larger B_r values and wider hysteresis curves
  - Soft materials are easily magnetized and demagnetized
  - Hard materials are difficult to demagnetize and make for good permanent magnets
  - The wider hysteresis curves of hard materials significantly increase the energy loss due to the magnetization-demagnetization cycles

- Since the relationship between $\vec{B}$ and $\vec{H}$ is no longer linear, for a ferromagnetic material we need to first determine its *operating condition* in order to determine its value of μ_r

Lecture 28, Mar 24, 2023

Magnetic Field Boundary Conditions

- Like the case with the electric field, we wish to find the boundary conditions for the tangential and normal $\vec{B}$ fields across a boundary, between medium μ_1 and medium μ_2, with the normal pointing from medium 2 to medium 1
- We can apply Gauss's law $\oiint_S \vec{B} \cdot \mathrm{d}\vec{s} = 0$ to an infinitely short cylinder right on the boundary, we can conclude that $B_{n1} - B_{n2} = 0 \implies B_{n1} = B_{n2}$
  - In terms of magnetic field intensity, $\vec{B} = \mu_r \mu_0 \vec{H} \implies \mu_{r1} H_{n1} = \mu_{r2} H_{n2}$
- For the tangential fields we can use an Amperian loop with width ΔL right on the boundary, so $\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = H_{t2}\Delta L - H_{t1}\Delta L = I_{enc} = J_s \Delta L$ where J_s is the surface current density on the boundary
  - This gives us $H_{t2} - H_{t1} = J_s$ or more formally $\hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$

Summary

Boundary conditions for magnetic fields across two mediums (with surface normal pointing from material 2 to material 1): For the normal component:

$$B_{n1} = B_{n2} \implies \mu_{r1} H_{n1} = \mu_{r2} H_{n2}$$

For the tangential component:

$$H_{t2} - H_{t1} = J_s$$

for a J_s normal to the tangential component, or

$$\hat{n}_2 \times (\hat{H}_1 - \hat{H}_2) = \vec{J}_s$$

- Due to this, as the field travels from a material with a low μ to one with high μ, it will be bent towards the surface (tangential component becomes larger)
  - For ferromagnetic materials μ_r can be really high, so the field becomes essentially entirely tangential
  - This can be used to perform magnetic shielding

Lecture 29, Mar 27, 2023

Magnetic Circuits: Example

- Example: toroid with N windings of wire carrying current I_0, with a piece cut out from it; what is $\vec{B}$ in that gap?
  - Boundary conditions have to be satisfied at the gap
  - Ignoring fringing effects, we would only have a B_n, so since B_n stays the same across a boundary, $\vec{B}$ also stays the same
  - If the current creates a field with intensity B_0, then $H_{\text{core}} = \dfrac{B_0}{\mu_{rc}\mu_0}, H_{\text{air}} = \dfrac{B}{\mu_0}$
    - Typically $H_{\text{core}} \ll H_{\text{air}}$
  - Apply Ampere's law with a contour aligned with the field: $\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = H_{\text{core}} L_{\text{core}} + H_{\text{air}} L_{\text{air}} = N I_0$
    - Since we have a toroid, we take L to be the mean distances
  - This gives $B_0 = \dfrac{N I_0}{\frac{L_c}{\mu_r \mu_0} + \frac{L_g}{\mu_0}}$

- To get flux, approximate as $\Phi = \iint_S \vec{B} \cdot \mathrm{d}\vec{s} = B_0 S = \dfrac{NI_0}{\frac{L_c}{\mu_r \mu_0 S} + \frac{L_g}{\mu_0 S}}$

- We can interpret NI_0 as a "voltage" of sorts, and the terms $\dfrac{L_c}{\mu_r \mu_0 S}$ and $\dfrac{L_g}{\mu_0 S}$ to be like "resistances"; this way we can think of this as a magnetic circuit, with "current" being the flux
- NI_0 is V_{mmf}, or the *magnetomotive force* (MMF); $\dfrac{L_c}{\mu_r \mu_0 S}$ is R, or the *reluctance*
  - MMF is the driving force in the same way voltage (electromotive force, EMF) is the driving force in an electric circuit
  - Reluctances resist the flux
  - In this case the reluctance of the core is much smaller than the reluctance of the air gap

Lecture 30, Mar 29, 2023

Magnetic Circuits

- In a magnetic circuits, sources are MMF, $V_m = NI_0$ where I_0 is the current and N is the number of loops
  - The direction of the winding determines the direction of the source
- Reluctance $R = \dfrac{L}{\mu S}$ is analogous to resistance in an electric circuit
- The magnetic flux Φ_i through the cross-section is analogous to electric current
- Permeability μ is analogous to conductivity σ in an electric circuit
- The usual laws of circuits apply:
  - $\sum_j V_{mj} = \sum_k R_k \Phi_k$ for a loop, like KVL
  - $\sum_i \Phi_i = 0$ for a node, like KCL
- Using the normal circuit analysis techniques (nodal, mesh analysis), we can find all fluxes
- To find the field, we assume a constant B in each cross section, so $B = \dfrac{\Phi}{S}$

Self and Mutual Inductance

- The field created by a current can cause fluxes in its own loop, or other loops

Definition

Inductance is defined as the amount of flux produced by a source per unit source,

$$L = \frac{\Lambda}{I} = \frac{N\Phi}{I}$$

where $\Lambda = N\Phi$ is the *flux linkage*; it is the dual of capacitance

Both self and mutual inductance exist, with

$$L_{ab} = \frac{N_b \Phi_{ab}}{I_a}$$

denoting the inductance in b caused by a

- Mutual inductances $L_{12} = L_{21}$
- Example: self-inductance of a toroid, magnetic core with N cores carrying current I_0
  - First find the current, from there find the field, then flux, then inductance

- Using Ampere's law with a contour aligned with the field, $2\pi r B = \mu_r \mu_0 N_1 I_1 \implies \vec{B} = \frac{\mu_r \mu_0 N_1 I_1}{2\pi r}\hat{a}_\phi$
- Integrate across the cross-section, $\Phi_{11} = \iint \vec{B}_1 \cdot \mathrm{d}\vec{s}_1 = \int_0^h \int_a^b \frac{\mu_r \mu_0 N_1 I_1 \hat{a}_\phi}{2\pi r} \cdot \hat{a}_\phi \,\mathrm{d}r\,\mathrm{d}z$
- Result: $L_{11} = \frac{\mu_r \mu_0 N_1^2 h}{2\pi} \ln \frac{b}{a}$
- Notice:
  * More turns directly leads to greater inductance
  * Greater area also leads to greater inductance
  * This only depends on geometry and material, never the current, etc

Lecture 31, Mar 31, 2023

Mutual Inductance Example

- Mutual inductance is usually denoted $L_{12} = M = L_{21}$; the fact that the mutual inductances go both ways is a result of the *reciprocity* rule
- Example: small circular loop of radius a, a distance d from an infinite wire carrying I_1; what is the mutual inductance of 1 due to 2, L_{21}?
  - We can instead find L_{12} since we don't know how to find the flux through an infinitely long wire
  - Approximate flux from the infinite wire as $B_1 \approx \frac{\mu_0 I_1}{2\pi d}$ (i.e. take the field at the centre of the circle), because the loop is small
  - $\Phi_{12} \approx \frac{\mu_0 I_1}{2\pi d}(\pi a^2)$
  - $L_{12} = L_{21} = M = \frac{N_2 \Phi_{12}}{I_1} = \frac{\mu_0 a^2}{2d}$

Magnetic Energy

- Like how electric potential energy is the energy it took to build up a collection of charges, magnetic potential energy is the energy it took to create a current distribution
- For free current distributions, this is due to Lenz's law – the field will oppose a change in current
- For bound current distributions, this is the energy required to align the magnetic dipoles within the material

Definition

The stored *magnetic potential energy* of a current distribution is

$$W_m = \frac{1}{2}\iiint_v \vec{B} \cdot \vec{H}\,\mathrm{d}v = \frac{1}{2}\iiint_v \mu_r \mu_0 |\vec{H}|^2\,\mathrm{d}v$$

The *magnetic energy density* is then

$$w_m = \frac{1}{2}\vec{B} \cdot \vec{H}$$

- Example: stored magnetic energy within an infinitely long solenoid
  - $H = nI, B = \mu_0 \mu_r n I$ by Ampere's law
  - $W_m = \frac{1}{2}\iiint_v \mu_r \mu_0 |\vec{H}|^2\,\mathrm{d}v = \frac{1}{2}\mu_r \mu_0 (\pi a^2 l)(n^2 I^2) = \frac{\mu_r \mu_0 \pi a^2 N^2 I^2}{2l}$
- The energy stored in an inductive element is $W_m = \frac{1}{2}LI^2$, which holds here as well
  - Often inductance is easier to find by first finding the energy and then solving for L

- Example: energy storage in coupled circular toroids – what is their self inductance, mutual inductance, and the energy stored?
  - First find B_1, B_2 from Ampere's law, with a loop concentric to the toroids going through them
  - We will approximate $B_1 = \frac{N_1 I_1 \mu_r \mu_0}{2\pi r_0}$ by assuming a constant B through the cross section (so the expression isn't a nightmare)
  - $B_1 = \frac{N_2 I_2 \mu_r \mu_0}{2\pi r_0}$
  - We can write the energy as $W_m = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + \frac{1}{2}L_{12}I_1 I_2 + \frac{1}{2}L_{21}I_1 I_2$
    * The first 2 terms are the self-energies; the second 2 terms are the mutual energies
  - To find L_{11}, L_{22} we first find W_m for the toroids
  - To find L_{12}, we integrate $\vec{B}_1 \cdot \vec{H}_2$, over the volume of the outer toroid
  - The volumes are chosen to be everywhere the field exists – in this case, we only consider the space in the toroids, since by Ampere's law the fields are zero outside them

Lecture 32, Apr 3, 2023

Time-Varying Fields: Overview

- So far we've discussed only static charges and steady currents; for these cases, electricity and magnetism are separate entities
- With time-varying charges and currents, electricity and magnetism are now related by Maxwell's equations:
  - $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
  - $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
  - $\vec{\nabla} \cdot \vec{D} = \rho_v$
  - $\vec{\nabla} \cdot \vec{B} = 0$
- Now changes in the electric field induce changes in the magnetic field and vice versa, which allows electromagnetic waves

Faraday's and Lenz's Laws

- A changing magnetic flux causes a current to flow in a closed loop; this means an electromotive force (EMF) is created
- Faraday's law states that the EMF induced in a circuit is directly proportional to the time rate of change of the magnetic flux linking that circuit
- The EMF is the amount of work done per unit charge, or $V_{emf} = \oint_C \frac{\vec{F}_e \cdot \mathrm{d}\vec{l}}{q} = \oint_C \vec{E} \cdot \mathrm{d}\vec{l}$
  - Because $\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$, an EMF can be caused by an electric force, a magnetic force, or a combination of both
  - Notice since the electric field is conservative in electrostatics, the EMF is zero without time-varying fields

Equation

Faraday's Law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \iff V_{emf} = \oint_C \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \mathrm{d}\vec{s} = -\frac{\partial \Phi}{\partial t}$$

- Electric fields induced by magnetic flux changes are not conservative, which is how EMF can be zero

- If the loop is closed, then the EMF will cause a current to flow, $I_{emf} = \frac{V_{emf}}{R}$
  - Lenz's law: The direction of this current is such that the magnetic field it produces opposes the original change in the magnetic field
  - Lenz's law is why there is a negative sign on Faraday's law
  - The field is not opposing the field, but opposing the change (e.g. if the field is up but decreasing, the induced current produces a field that still points up, to compensate the decrease)
- Lenz's law is a statement of the conservation of energy; if the induced current flowed the other way, it would lead to a positive feedback loop and violate conservation of energy
- With multiple turns in the loop, $V_{emf} = -N\frac{\partial \Phi}{\partial t} = -N\frac{\partial}{\partial t}\iint_S \vec{B} \cdot \mathrm{d}\vec{s}$ since every turn experiences the same EMF
- To cause this change in the flux, we could either change $\vec{B}$ itself or change the surface S (e.g. expand/shrink, change in orientation)
  - Changes in flux caused by changes in $\vec{B}$ causes an EMF known as the *transformer EMF*
    - This is created by the induced electric force
  - Changes in the flux caused by changes in S is known as a *motional EMF*
    - This is caused by moving charges in the presence of a $\vec{B}$
- Example: transformer EMF: find induced V_{emf} in a torus, if $I(t) = I_0 \cos(\omega t)$ passes through the centre, N turns of wire, inner diameter a, outer diameter b, height c
  - $\vec{B}(t) = \frac{\mu I(t)}{2\pi r}$
  - $\Phi(t) = \iint_S \vec{B}(t) \cdot \mathrm{d}\vec{s} = \int_a^b \int_{-c}^0 \frac{\mu I(t)}{2\pi r}\,\mathrm{d}z\,\mathrm{d}r = \frac{\mu c I(t)}{2\pi}\ln\frac{b}{a} = \frac{\mu c \ln\left(\frac{b}{a}\right)}{2\pi} I_0 \cos(\omega t)$
  - $V_{emf} = -N\frac{\partial \Phi}{\partial t} = \frac{N\mu c \ln\left(\frac{b}{a}\right) I_0 \omega}{2\pi}\sin(\omega t)$
  - Notice: $V_{emf} = V_0 \sin(\omega t) = L\frac{\mathrm{d}I}{\mathrm{d}t}$

Lecture 33, Apr 5, 2023

Transformer EMFs

- In the case of transformer EMFs the surface's relationship with $\vec{B}$ says constant
- $V_{emf} = -N\frac{\partial \Phi}{\partial t} = -N\iint_S \frac{\partial}{\partial t}\vec{B} \cdot \mathrm{d}\vec{s}$
  - The Φ is total flux flowing through the loop; this includes both applied and the flux caused by the induced EMF
  - The induced EMF/current, through self inductance, will also cause its own EMF
  - $V_{emf} = -\frac{\partial \Phi_{net}}{\partial t} = -\frac{\partial}{\partial t}(\Phi_{app} + \Phi_{ind})$
- We can account for the effect of the induced current by including an inductor, $V = L\frac{\mathrm{d}I_{ind}}{\mathrm{d}t}$, so $V_{emf} = RI_{ind} + L\frac{\mathrm{d}I_{ind}}{\mathrm{d}t}$
- In general to actually find the induced current we need to solve a differential equation; but often we will just ignore the effects of the induced current, so $I_{ind} \approx \frac{V_{emf}}{R}$
  - This is a reasonable assumption when the self-inductance L is small or $\frac{\mathrm{d}I}{\mathrm{d}t}$ is small

Important

At lower frequencies, self-inductance can be ignored; however, at higher frequencies, self-inductance must be accounted for through differential equations as they can have major impacts on overall behaviour

Lecture 34, Apr 10, 2023

Example: Transformers

- From a magnetic circuit perspective we have 2 sources, $N_1 i_1$ and $N_2 i_2$, and a resistance $R_c = \dfrac{l_c}{\mu_r \mu_0 S}$
- We obtain $N_1 i_1 - N_2 i_2 = R_c \Phi_{tot}$ by KVL
- Making the approximation that $\mu_r \to \infty$ we get the relation for an ideal transformer: $\dfrac{i_1}{i_2} = \dfrac{N_2}{N_1}$
- In reality, the core will not be perfect and there will be flux leakage so this relation is not exact
- With higher frequencies this becomes more noticeable, and we also see a phase shift in the output
  - We have losses in the wire resistances, hysteresis loss, eddy current losses, self-inductances (which become problematic at higher frequencies), etc
- The transformer becomes less ideal as frequency increases, with reduced output amplitude and increased phase shift
- Idealized formulas work fine for power distribution systems which are typically 60 Hz, but at higher frequencies approximations fall apart

Eddy Currents

- Changing $\vec{B}$ leads to a changing $\vec{E}$, which will induce a $\vec{J}$ in a conducting material
- These are referred to as "eddy currents" since they circulate
- Consider an applied field $\vec{B}(t) = B_0 \cos(\omega t)\hat{a}_z$ on a cylinder made of a lossy material with conductivity σ
  - $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} = B_0 \omega \sin(\omega t)\hat{a}_z$
  - $\dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(rE_\phi) - \dfrac{\partial E_r}{\partial \phi}\right) = B_0 \omega \sin(\omega t)\hat{a}_z$
  - We can deduce $\dfrac{\partial E_r}{\partial \phi} = 0$ and so $\vec{E} = \dfrac{B_0 \omega r \sin(\omega t)}{2}\hat{a}_\phi$
  - This gives us eddy currents $\vec{J} = \dfrac{\sigma B_0 \omega r \sin(\omega t)}{2}\hat{a}_\phi$
- These eddy currents generate fields of their own that oppose the original field; this causes the effect of a magnet falling slower in a metallic tube
  - This can be used in applications such as frictionless braking
  - However one disadvantage is that the braking force reduces as the speed slows, since the braking force is proportional to the rate of change of the field

Lecture 35, Apr 12, 2023

Motional EMFs

- We can think of it in 2 perspectives, either due to changing magnetic flux $V_{emf} = -\dfrac{\partial \Phi}{\partial t}$ or from a perspective of magnetic force $\vec{F}_m = q\vec{u} \times \vec{B}$
- Electrons in a moving conductor will have some velocity due to the movement of the conductor as a whole, so in the presence of a field they experience a force, causing an EMF
- This gives us $V_{emf} = \oint_C \dfrac{\vec{F}_m \cdot \mathrm{d}\vec{l}}{q} = \oint_C (\vec{u} \times \vec{B}) \cdot \mathrm{d}\vec{l}$
  - The $\vec{u}$ only exists on the moving parts of the conductor, so we can ignore any stationary parts
  - Note electrons have negative charge
- Application note: if we have a loop rotating in a uniform magnetic field, then $\vec{B} \cdot \mathrm{d}\vec{S}$ will vary as $\cos(\omega t)$, so we will produce a sinusoidal AC voltage
  - $V_{emf} = B_0 S \omega \sin(\omega t)$
  - Note the amplitude scales directly with frequency, and the output frequency is the same as the

input frequency of the turning
  - What if we used an AC current to produce the field in the first place?
    * The EMF will now be the total EMF, a combination of both the transformer and motional EMFs
    * $V_{emf} = -N\frac{\partial\Phi}{\partial t} = -N\frac{\partial}{\partial t}\iint B_0\sin(\omega t)\hat{a}_z \cdot \mathrm{d}\vec{s}$
    * The dot product introduces another cosine, so we end up integrating the product of a sine and cosine
    * This gives us a resulting voltage that varies with a frequency of $2\omega t$

Lecture 36, Apr 14, 2023

Maxwell's Contribution – Displacement Current

- Ampere's law breaks down when we consider a simple circuit with a capacitor and a surface through the middle of the capacitor
- A new type of "current" has to be considered – the *displacement current*

Definition

The displacement current is defined as

$$\vec{J}_d = \frac{\partial\vec{D}}{\partial t}$$

with Ampere's law becoming

$$\vec{\nabla}\times\vec{H} = \vec{J} + \vec{J}_d$$

or in the integral form:

$$\oint_C \vec{H}\cdot\mathrm{d}\vec{l} = \iint_S \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot\mathrm{d}\vec{s} = I_c + I_d$$

- The displacement current was initially an educated guess by Maxwell which was experimentally verified later
- Unlike $\vec{J}$, the displacement current $\vec{J}_d$ is not due to the movement of charges
- This now allows the existence of electromagnetic waves – a change in $\vec{D}$ induces a change in $\vec{H}$ by Ampere's law, and a change in $\vec{H}$ induces a change in $\vec{D}$ by Faraday's law