Lecture 8, Oct 4, 2023

Rules of Thumb for ODEs

- Consider the ODE: $\dot{x} = -\frac{1}{\tau}x, x(0) = x_0$
  - The solution is $x_0 e^{-\frac{1}{\tau}t}$
  - τ is the *time constant* of the system
  - Numerical stability for forward Euler requires $\left|1 - \frac{h}{\tau}\right| < 1 \implies 0 \leq h \leq 2\tau$
  - So for the system to be stable, h depends on τ
  - In general, how can we determine the simulation parameters such as simulation step size h and simulation time T for an arbitrary ODE?
- Consider $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ with no input
  - Depending on the eigenvalues the solution can behave differently shown in the diagram below

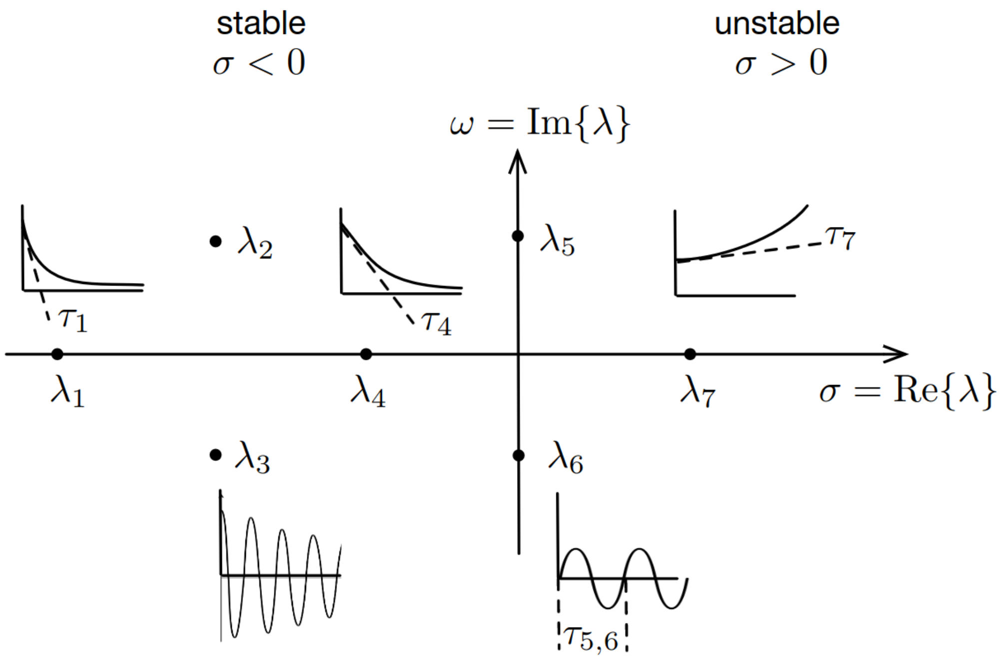

Figure 1: Behaviour of the ODE is determined by eigenvalues.

- We can assign a time constant based on the eigenvalues
  - Purely real eigenvalue: $\tau_i = \frac{1}{|\lambda_i|}$
    - In this case τ describes the rate of decay of the system
  - Purely imaginary: $\tau_i = \frac{2\pi}{|\lambda_i|}$
    - In this case τ is the period of the oscillation; faster oscillations means smaller τ
  - For a complex eigenvalue, we will consider both: $\tau_i = \left\{ \frac{1}{\mathrm{Re}\{\lambda_i\}}, \frac{2\pi}{\mathrm{Im}\{\lambda_i\}} \right\}$
- We define the *stiffness* of the system as $\gamma = \frac{\tau_{max}}{\tau_{min}}$ as the ratio between the maximum and minimum time constants; the ODE is considered *stiff* when $\tau > 10^3$
  - We only care about the max and min time constants since if we can compute the fastest and slowest behaviours of our system, we can compute anything in-between
- General rule of thumb:
  - Take the simulation time to be $T = 5\tau_{max}$ if the system is stable

    * If the system is unstable, stop when a component of x exceeds a threshold
- Take the step size to be $h = \min\left\{ \frac{\tau_{min}}{10}, \frac{T}{200} \right\}$
    * This gives the number of steps as $k = \frac{T}{h} = \max\{ 50\gamma, 200 \}$
- For plotting, take the step size to be $H = \frac{T}{200} = \frac{\tau_{max}}{40}$, since we do not need to plot every step
- For stiff ODEs, fixed-step solvers are generally very expensive, so variable step-size solvers with an initial step size of $\frac{\tau_{min}}{10}$ should be used
    * In MATLAB, solvers ending with "s" are good for stiff systems, e.g. `ode23s`, `ode15s`