Lecture 7, Sep 28, 2023

Stability for Nonlinear Systems – Lyapunov's Method

- In general a nonlinear model is characterized by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$
- One approach is locally linearizing using the Jacobian around a particular state and control input
  - $\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ where $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_d$ and $\boldsymbol{x}_d$ is the set point
  - $\boldsymbol{A} = \left.\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^T}\right|_{\boldsymbol{x}=\boldsymbol{x}_d}, \boldsymbol{B} = \left.\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}^T}\right|_{\boldsymbol{u}=\boldsymbol{0}}$
  - With this we can apply the normal feedback methods with $\boldsymbol{u} = -\boldsymbol{F}\Delta\boldsymbol{x} \implies \Delta\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})\Delta\boldsymbol{x}$ and choose $\boldsymbol{F}$ appropriately to put the poles in the left-half plane
  - Because this a local approximation, it will not work when the state is significantly different from the linearization point
- Another approach is *gain scheduling*, where we design a set of gains for a variety of different set points of the nonlinear system (i.e. "scheduling" the gains according to where you are in state space)
  - However this requires a lot of work and more importantly cannot guarantee stability
- To guarantee stability for a nonlinear system, we can use Lyapunov's method

Definition

The solution $\boldsymbol{x}(t; \boldsymbol{x}_0, t_0)$ to the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is said to be *stable* in the Lyapunov sense (aka L-stable) if

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ s.t. } \|\Delta\boldsymbol{x}_0\| < \delta \implies \forall t > t_0, \|\Delta\boldsymbol{x}\| < \varepsilon$$

$\boldsymbol{x}$ is *asympotitcally stable* if $\lim_{t\to\infty} \|\Delta\boldsymbol{x}\| = 0$; *exponential stability* further requires that $\|\Delta\boldsymbol{x}\|$ decreases exponentially.

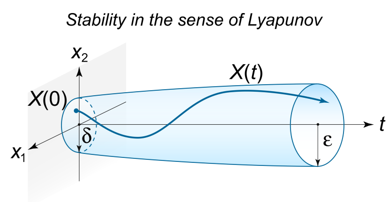

Figure 1: Illustration of Lyapunov stability.

Definition

A function $v(\boldsymbol{x})$ is *positive-definite* if

$$\forall\boldsymbol{x} \neq \boldsymbol{0}, v(\boldsymbol{x}) > 0 \text{ and } v(\boldsymbol{0}) = 0$$

and *negative-definite* if

$$\forall\boldsymbol{x} \neq \boldsymbol{0}, v(\boldsymbol{x}) < 0 \text{ and } v(\boldsymbol{0}) = 0$$

$v(\boldsymbol{x})$ is positive/negative-*semidefinite* if $v(\boldsymbol{x}) \geq 0/v(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x}$.

Theorem

Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(x)$ with an equilibrium at $\boldsymbol{x} = \boldsymbol{0}$; if we can find a positive-definite $v(\boldsymbol{x})$, and $\dot{v}(\boldsymbol{x})$ is negative-semidefinite, then $\boldsymbol{x} = \boldsymbol{0}$ is stable. If $\dot{v}(\boldsymbol{x})$ is negative-definite, then $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable. Note that

$$\dot{v} = \frac{\partial v}{\partial \boldsymbol{x}^T}\dot{\boldsymbol{x}} = \frac{\partial v}{\partial \boldsymbol{x}^T}\boldsymbol{f}(\boldsymbol{x})$$

This function $v(\boldsymbol{x})$ is known as a *Lyapunov function.*

- $v(\boldsymbol{x})$ can be thought of as a potential energy surface; since $\dot{v}(\boldsymbol{x})$ is negative-(semi)definite, we always go down the surface, and since $v(\boldsymbol{x})$ is positive definite, we can't go down lower than 0, which is the location of the equilibrium
  - If $\dot{v}(\boldsymbol{x})$ is merely negative-semidefinite, we can get "stuck" before reaching the equilibrium (e.g. in a local minimum), but the solution is still stable
- Just because you can't find a Lyapunov function doesn't mean the system is unstable!
- However we can invert the result and find a positive-definite $\dot{v}(x)$, which would mean the system is unstable
- Example: $\dot{x}_1 = x_2 + \alpha x_1(x_1^2 + x_2^2), \dot{x}_2 = -x_1 + \alpha x_2(x_1^2 + x_2^2)$
  - We can take $v(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ which is clearly positive-definite
  - $\dot{v} = x_1\dot{x}_1 + x_2\dot{x}_2 = \alpha(x_1^2 + x_2^2)$
  - Therefore the system is asymptotically stable if $\alpha < 0$ or merely stable if $\alpha \leq 0$
  - For this example we can also say that if $\alpha > 0$, the system is unstable since $\dot{v}$ is positive-definite

Theorem

Lasalle's extension: If $\dot{v}$ is only negative-semidefinite, but the only solution to $\dot{v}(\boldsymbol{x}) = 0$ and $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is $\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

- The idea is that Lyapunov's theorem considers all $\boldsymbol{x}$, but we only care about the ones that satisfy the equation of motion; so if $\dot{v}(\boldsymbol{x}) = \boldsymbol{0}$ is only possible at $\boldsymbol{x} = 0$ if the equation of motion must be satisfied, then the system is still asymptotically stable
  - Usually when Lasalle's extension applies, we have a $\dot{v}$ that is zero when only some of the x_i are zero, but does not require all of them to be zero; so if satisfying $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with these $x_i = 0$ requires all the other coordinates to be zero, then $\boldsymbol{x} = \boldsymbol{0}$ is still asymptotically stable

Example: Feedback Tracking Problem

- Consider a robot with a unicycle model; we want to track a path $(x_d(t), y_d(t), \theta_d(t))$
  - $\begin{bmatrix}\dot{x}_d \\ \dot{y}_d \\ \dot{\theta}_d\end{bmatrix} = \begin{bmatrix}u_d \cos\theta_d \\ u_d \sin\theta_d \\ \omega_d\end{bmatrix}$
  - u_d, ω_d are the control inputs that will get us exactly to the setpoint in a perfect world; however since we might have disturbances we need feedback control
- We will do a coordinate transform into the robot coordinate system with axes ξ parallel to the robot and η perpendicular to it
  - $\boldsymbol{\xi} = \begin{bmatrix}\xi \\ \eta \\ \theta\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x \\ y \\ \theta\end{bmatrix}$ corresponding to a rotation about the third axis
  - $\dot{\xi} = \dot{x}\cos\theta + \dot{y}\sin\theta + (-x\sin\theta + y\cos\theta)\dot{\theta} = \dot{x}\cos\theta + \dot{y}\sin\theta + \eta\dot{\theta}$
  - $\dot{\eta} = -\dot{x}\sin\theta + \dot{y}\cos\theta + (x\cos\theta + y\sin\theta)\dot{\theta} = -\dot{x}\sin\theta + \dot{y}\cos\theta + \xi\dot{\theta}$
- We can make the same transformation for the desired coordinates $(x_d, y_d, \theta_d) \rightarrow (\xi_d, \eta_d, \theta_d)$
  - $\xi_d = x_d\cos\theta + y_d\sin\theta$

- $\eta_d = -x_d \sin\theta + y_d \cos\theta$
    - $\dot{\xi}_d = \dot{x}_d \cos\theta + \dot{y}_d \sin\theta + \eta_d \dot{\theta} = u_d \cos(\theta - \theta_d) + \eta_d \omega$
    - $\dot{\eta}_d = -\dot{x}_d \sin\theta + \dot{y}_d \cos\theta + \xi_d \dot{\theta} = u_d \sin(\theta - \theta_d) - \xi_d \omega$
- Let the error $e_x = \xi - \xi_d, e_y = \eta - \eta_d, e_\theta = \theta - \theta_d$
    - We've converted a tracking problem to a regulator problem
    - We want to send all these error terms to zero
- The error derivatives are $\dot{\boldsymbol{e}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} u_d \cos e_\theta + u + e_y \omega \\ u_d \sin e_\theta - e_x \omega \\ \omega - \omega_d \end{bmatrix}$
- Our control algorithm will be $u = -k_x e_x - u_d \cos e_\theta, \omega = \omega_d - k_\theta \sin e_\theta - u_d e_y$
    - In the end we get a nonlinear function $\dot{\boldsymbol{e}} = \boldsymbol{\Phi}(\boldsymbol{e})$
- Choose a candidate Lyapunov function $v(e_x, e_y, e_\theta) = \frac{1}{2}(e_x^2 + e_y^2) + (1 - \cos e_\theta)$
    - Notice that these terms are energy-like: the $\frac{1}{2}(e_x^2 + e_y^2)$ is spring energy in 2D and $1 - \cos e_\theta$ is the energy of a pendulum; this is usually a good guide to selecting candidate Lyapunov functions
    - $\dot{v} = -k_x e_x^2 - k_\theta \sin^2 e_\theta$
    - If $k_x, k_\theta > 0$, $\dot{v}$ is negative definite but only with respect to e_x and e_θ; this means it is negative-semidefinite
    - Lyapunov's theorem alone tells us only that the system is stable, but not necessarily asymptotically so
    - We can try applying Lasalle's extension, if we can show that $e_x = e_\theta = 0 \implies e_y = 0$ in order to satisfy the equation of motion
        * If we substitute back in $e_x = e_\theta = 0$ (and $\dot{e}_x = \dot{e}_\theta = 0$) we can prove that $e_y = 0$, so by Lasalle's extension this system is asymptotically stable