Lecture 4, Sep 19, 2023

Motion of Robots

- We will concentrate primarily on rolling, i.e. robots with wheels
- Steering models:
  - Bicycle steering: traction wheel in rear, steering wheel in front
    - Tricycle steering: the rear wheel is unpowered, while the front steering wheel is powered
  - Differential steering: two independently controlled wheels, vary speed to get desired curvature
    - Tripod differential steering: differential steering with an unpowered omnidirectional wheel in front for support
    - Skid-steering: each side is controlled together
  - Directed differential steering: differential steering aided by a steering wheel
  - Ackermann steering: two differentially operated wheels at the back and two connected steering wheels at the front
- Wheels can also be compound:
  - Mecanum wheels: wheels with angled rollers on the surface
    - Through moving the wheels in different directions, motion in any of the 4 directions or rotation can be achieved
    - The wheels produce forces in diagonal directions; combining these forces results in a net force in the desired direction
  - Omni wheels: like the Mecanum wheel, but the rollers are perpendicular instead of diagonal

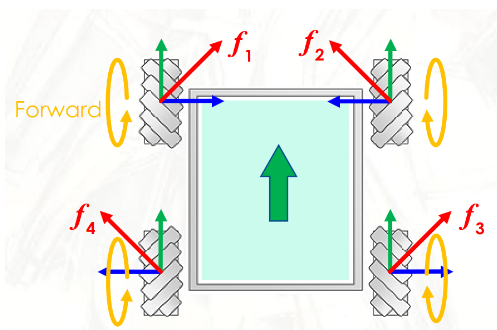

Figure 1: Mecanum wheels in translation.

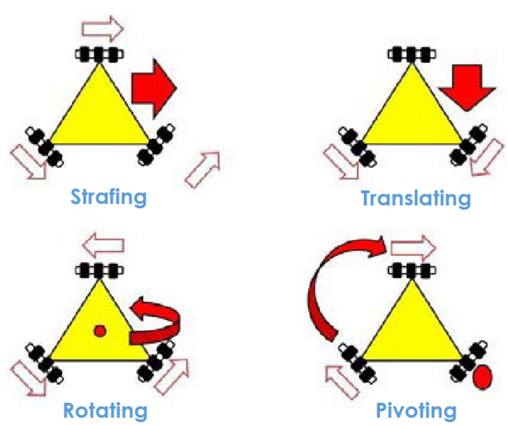

Figure 2: Omni wheels used on a vehicle.

- Holonomic constraint: a vehicle is holonomic if the vehicle's geometry does not constrain its motion (i.e. it can move in any direction, regardless of which direction it's facing)

- Mecanum and omni drive are holonomic, but Ackermann is not
- We will formally define this in AER301

Kinematical Models of Motion

- The *pose* of a robot is its position and orientation
  - For now we will work in 2D, with position being x, y and orientation being θ, so the pose is $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$
- For a simple unicycle model, θ is the angle of the wheel and x, y are the position of the wheel on the ground
  - $\dot{x} = v\cos\theta, \dot{y} = v\sin\theta, \dot{\theta} = \omega$
  - $\boldsymbol{x} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$
  - $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \implies \dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{u}$
  - $\boldsymbol{x}$ is the state and $\boldsymbol{u}$ is the system input

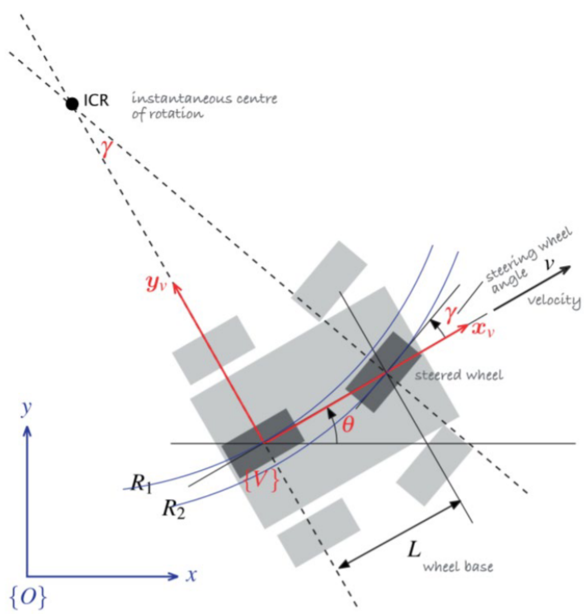

Figure 3: Bicycle model derivation.

- For a bicycle model, refer to the diagram above
  - We will fix our reference point to the rear wheel; θ is the angle the rear wheel makes with the horizontal axis
  - $\dot{x} = v\cos\theta, \dot{y} = v\sin\theta$ as usual
  - To find $\dot{\theta}$, we extend a perpendicular line from the wheels to intersect at the instantaneous center of rotation
  - $v = R_1\dot{\theta}, \dfrac{l}{R_1} = \tan\gamma \implies \dot{\theta} = \dfrac{v}{l}\tan\gamma$
  - The control inputs are v and γ
  - Notice that this model is now nonlinear due to the tangent on γ and multiplication by v
- For differential steering our control inputs are $\dot{\gamma}_r, \dot{\gamma}_l$, which are the rotational rates of the two wheels

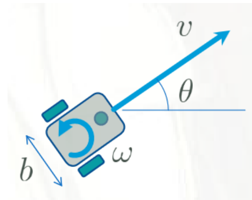

Figure 4: Differential steering derivation.

- $v = \dfrac{r(\dot{\varphi}_r + \dot{\varphi}_l)}{2}$ (velocity is simply the average)
- $\omega = \dfrac{r(\dot{\varphi}_r - \dot{\varphi}_l)}{b}$
- We can then put this into the unicycle model to obtain the final model
- $\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} r \cos\theta & \frac{1}{2} r \cos\theta \\ \frac{1}{2} r \sin\theta & \frac{1}{2} r \sin\theta \\ \frac{r}{b} & -\frac{r}{b} \end{bmatrix} \begin{bmatrix} \dot{\varphi}_r \\ \dot{\varphi}_l \end{bmatrix}$

Wheel Models

- To generalize our motion models, we want to derive the general model for a standard wheel

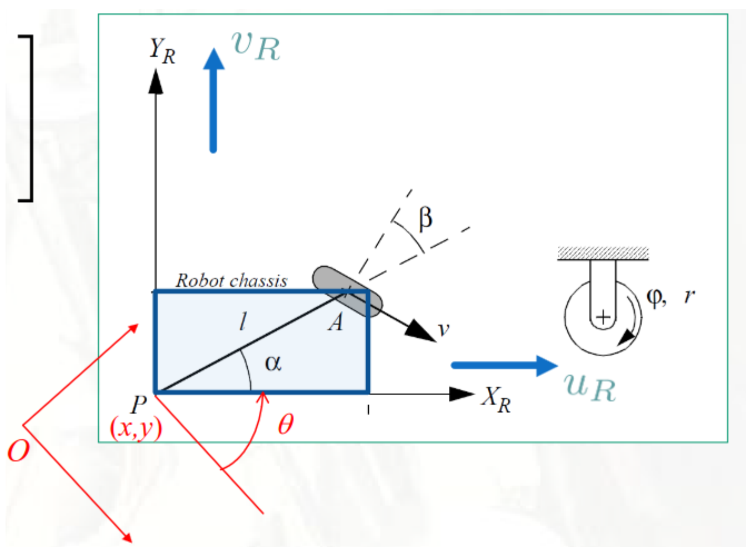

Figure 5: Derivation of the standard wheel model.

- Let $\underset{\rightarrow}{\mathcal{F}}_g$ be the global reference frame, let $\underset{\rightarrow}{\mathcal{F}}_r$ be the vehicle reference frame and $\underset{\rightarrow}{\mathcal{F}}_w$ be the wheel reference frame
  - For all 3 frames the 3rd vector points up
  - $\underset{\rightarrow}{r}_1$ is parallel to the vehicle and $\underset{\rightarrow}{r}_2$ is normal to it
  - $\underset{\rightarrow}{w}_1$ is normal to the wheel and $\underset{\rightarrow}{w}_2$ is parallel to it
  - Since everything is in the same plane, we have $\boldsymbol{C}_{rg} = \boldsymbol{C}_3(\theta), \boldsymbol{C}_{wr} = \boldsymbol{C}_3(\alpha + \beta)$
- We'll use the notation that $\underset{\rightarrow}{\rho}^{XY}$ being the position of point X measured in frame Y (if Y is omitted, it is the global frame)
- For any wheel, the kinematics of the vehicle is defined by the constraints of the wheel
  - We will assume that the wheel does not slip, so it cannot move in the direction of $\underset{\rightarrow}{w}_1$
    * This imposes a constraint $\underset{\rightarrow}{v}^A \cdot \underset{\rightarrow}{w}_1 = 0$, that is, $\underset{\rightarrow}{v}^A$ has no velocity in the direction of $\underset{\rightarrow}{w}_1$

- The wheel can roll freely in the direction of $\underrightarrow{w}_2$
    * This means means that $\underrightarrow{v}^A \cdot \underrightarrow{w}_2 = -r\dot{\varphi}$
- We want $\underrightarrow{v}^A = \left.\frac{\mathrm{d}}{\mathrm{d}t}\underrightarrow{\rho}^A\right|_{\underrightarrow{\boldsymbol{\mathcal{F}}}_g} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\underrightarrow{\rho}^P\right|_{\underrightarrow{\boldsymbol{\mathcal{F}}}_g} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\underrightarrow{\rho}^{AP}\right|_{\underrightarrow{\boldsymbol{\mathcal{F}}}_g} = \underrightarrow{v}^P + \left.\frac{\mathrm{d}}{\mathrm{d}t}\underrightarrow{\rho}^{AP}\right|_{\underrightarrow{\boldsymbol{\mathcal{F}}}_r} + \underrightarrow{\omega}^{rg} \times \underrightarrow{\rho}^{AP}$
    - $\underrightarrow{v}^P = \underrightarrow{\boldsymbol{\mathcal{F}}}_g^T \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} = \underrightarrow{\boldsymbol{\mathcal{F}}}_r^T \boldsymbol{C}_{rg} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} = \underrightarrow{\boldsymbol{\mathcal{F}}}_r^T \begin{bmatrix} u_r \\ v_r \\ 0 \end{bmatrix}$
    - $\underrightarrow{\omega}^{rg}$ is the angular velocity of $\underrightarrow{\boldsymbol{\mathcal{F}}}_r$ with respect to $\underrightarrow{\boldsymbol{\mathcal{F}}}_g$ so it's simply $\underrightarrow{\boldsymbol{\mathcal{F}}}_r^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$
    - $\underrightarrow{\rho}^{AP} = \underrightarrow{\boldsymbol{\mathcal{F}}}_r^T \begin{bmatrix} l\cos\alpha \\ l\sin\alpha \\ 0 \end{bmatrix}$ so it has a derivative of 0
    - Therefore we can get $\underrightarrow{v}^A = \underrightarrow{v}^D + \underrightarrow{\omega}^{rg} \times \underrightarrow{\rho}^{AP}$ and then express it in frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_w^T$
        * This works out to be $\underrightarrow{\boldsymbol{\mathcal{F}}}_w^T \begin{bmatrix} u_R\cos(\alpha+\beta) + v_R\sin(\alpha+\beta) + l\dot{\theta}\sin\beta \\ -u_R\sin(\alpha+\beta) + v_R\cos(\alpha+\beta) + l\dot{\theta}\cos\beta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -r\dot{\varphi} \\ 0 \end{bmatrix}$ (due to the constraints) where u_R, v_R are the components of the robot's velocity along and normal to its frame
- The two equations $\begin{cases} u_R\cos(\alpha+\beta) + v_R\sin(\alpha+\beta) + l\dot{\theta}\sin\beta = 0 \\ -u_R\sin(\alpha+\beta) + v_R\cos(\alpha+\beta) + l\dot{\theta}\cos\beta = -r\dot{\varphi} \end{cases}$ define the wheel kinematics