Lecture 22, Nov 30, 2023

Manipulator Control

- Recall the manipulator dynamics: $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^f(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^g(\boldsymbol{q}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{f}^{ee} = \boldsymbol{u}(t)$
  - $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$ are the linear (in $\ddot{\boldsymbol{q}}$) terms
  - $\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are the nonlinear (in $\dot{\boldsymbol{q}}$) terms
- In general there are 3 approaches to control: independent joint control, computed torque control, and general feedback control

Independent Joint Control

- Assumes joints are independent; each joint is controlled by its own PID controller
- $u_k(t) = -K_{D,k}(\dot{q}_k - \dot{q}_{d,k}) - K_{P,k}(q_k - q_{d,k}) - K_{I,k}\int (q_k - q_{d,k})\,\mathrm{d}t$
  - $q_{d,k}(t)$ is the desired trajectory of joint k
- Most simple and most commonly used; does not take into account the system dynamics at all
- Since the system is highly nonlinear, there is no guarantee that this will work
- In practice gain scheduling might be used to improve results

Computed Torque Control

- Using the equations of motion, solve for the forces to effect the desired motion, and use PD control to correct for errors
- $\boldsymbol{u}(t) = \boldsymbol{M}(\boldsymbol{q})[\ddot{\boldsymbol{q}}_d - \boldsymbol{K}_D(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d) - \boldsymbol{K}_P(\boldsymbol{q} - \boldsymbol{q}_d)] + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^g(\boldsymbol{q})$
  - Note in the following discussion we will neglect friction and end-effector force terms
- This uses a combination of feedback and feedforward control
- This requires that we know all parts of the system dynamics fairly well
- We can show that this is asymptotically stable; substitute $\boldsymbol{u}(t)$ in the manipulator dynamics, then:
  - $\boldsymbol{M}(\boldsymbol{q})[\ddot{\boldsymbol{q}}_d - \boldsymbol{K}_D(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d) - \boldsymbol{K}_P(\boldsymbol{q} - \boldsymbol{q}_d)] + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^g(\boldsymbol{q}) = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^g(\boldsymbol{q})$
  - $\boldsymbol{M}(\boldsymbol{q})[(\dot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_d) - \boldsymbol{K}_D(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d) - \boldsymbol{K}_P(\boldsymbol{q} - \boldsymbol{q}_d)] = \boldsymbol{0}$
  - Since $\boldsymbol{M}$ is positive definite, this reduces to $\ddot{\boldsymbol{e}} + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} = \boldsymbol{0}$, where $\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}_d$
  - This is asymptotically stable if $\boldsymbol{K}_D, \boldsymbol{K}_D$ are both positive definite
- $\boldsymbol{K}_D, \boldsymbol{K}_P$ can be chosen to be e.g. diagonal matrices, in which case this would be similar to independent joint control, but with feedforward to take into account manipulator dynamics
  - If $\boldsymbol{K}_D = \mathrm{diag}[2\zeta_i\omega_i], \boldsymbol{K}_P = \mathrm{diag}[\omega_i^2]$, then the error equation is $\ddot{e}_i + 2\zeta_i\omega_i\dot{e}_i + \omega_i^2 e_i = 0$

General Feedback Control

- PD controller with some feedforward for gravity, but not inertia
- $\boldsymbol{u}(t) = -\boldsymbol{K}_D(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d) - \boldsymbol{K}_P(\boldsymbol{q} - \boldsymbol{q}_d) - \boldsymbol{f}^g(\boldsymbol{q})$
  - Notice that there is no $\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ term or inertia matrix
  - This requires much less knowledge of the system than the computed-torque approach
- Here we will analyze only the regulator problem (i.e. constant $\boldsymbol{q}_d$)
- We can prove that this is stable using Lyapunov theory:
  - Candidate Lyapunov function: $v(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{e}^T\boldsymbol{K}_P\boldsymbol{e}$
    - Since $\boldsymbol{M} > 0$, assuming $\boldsymbol{K}_P > 0$, this is clearly positive definite
  - $\dot{v}(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{K}_P\boldsymbol{e}$
    - From the equation of motion, $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = -\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} - \boldsymbol{K}_P\boldsymbol{e}$ (obtain by substituting in control policy)
    - $\dot{v}(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \dot{\boldsymbol{q}}^T\boldsymbol{K}_D\dot{\boldsymbol{q}}$
    - The last term is negative semi-definite, but what about the rest?

- $\frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \sum_{k=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}\dot{M}_{kj}\dot{q}_j - h_k\right)\dot{q}_k$
  * Recall $h_k = \sum_{j=1}^{n}\left(\dot{M}_{kj} - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\right)\dot{q}_j$
  * Therefore $\frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\dot{M}_{kj} - \sum_{i=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\right)\dot{q}_j\dot{q}_k$
  * Note $\dot{M}_{kj} = \sum_{i=1}^{n}\frac{\partial M_{kj}}{\partial q_i}\dot{q}_i$
  * Therefore $\frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{kj}}{\partial q_i} - \frac{\partial M_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j\dot{q}_k$
  * But $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j\dot{q}_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\dot{q}_k$ if we rename the indices
  * Therefore this entire term reduces to 0
- Hence $\dot{v}(\boldsymbol{e},\dot{\boldsymbol{q}}) = -\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\dot{\boldsymbol{q}}$
  * Provided $\boldsymbol{K}_D > 0$, this is negative definite with respect to $\dot{\boldsymbol{q}}$, but not $\boldsymbol{e}$
  * We need to use Lasalle's extension
- Consider the equation of motion: $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{K}_P\boldsymbol{e} = \boldsymbol{0}$
  * When $\ddot{\boldsymbol{q}} = \boldsymbol{0}$, we also have $\ddot{\boldsymbol{q}} = 0$, so the equation of motion reduces to $\boldsymbol{K}_P\boldsymbol{e} = \boldsymbol{0}$
  * Since $\boldsymbol{K}_P$ is positive definite, it is also full rank, so the only solution is $\boldsymbol{e} = \boldsymbol{0}$
  * Therefore when $\dot{v} = 0$, we are forced to have $\boldsymbol{e} = \boldsymbol{0}$, so Lasalle's extension applies
- Hence this system is asymptotically stable if $\boldsymbol{K}_P, \boldsymbol{K}_D > 0$