Lecture 21, Nov 28, 2023

Manipulator Dynamics

- For typical mobile robots, dynamics aren't all too important to their motion
- But for manipulator systems, we may need fast movements, making dynamics important
- We would like to derive the equations of motion for manipulators, using the Lagrangian formulation

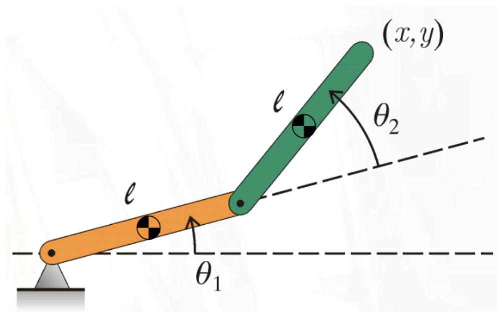

Figure 1: Two-link manipulator.

- Consider the 2-link manipulator with coordinates θ_1, θ_2, both links with length l, masses m_1, m_2, and moments of mass a bout joints c_1, c_2, I_1, I_2, and centres of mass at midlink
  - $T_1 = \frac{1}{2}m_1\left(\frac{1}{2}l\dot{\theta}_1\right)^2 + \frac{1}{2}I_{\otimes,1}\dot{\theta}_1^2 = \frac{1}{2}\left(I_{\otimes,1} + \frac{1}{4}m_1 l^2\right)\dot{\theta}_1^2 = \frac{1}{2}I_1\dot{\theta}_1^2$
    * Note we started with the kinetic energy relative to the centre of mass, where we have both a translational and a rotational component
    * This is equivalent to applying the parallel axis theorem
  - The speed of link 2's centre of mass is $v_{\otimes,2}^2 = (l\dot{\theta}_1)^2 + \left(\frac{1}{2}l(\dot{\theta}_1 + \dot{\theta}_2)\right)^2 + l^2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2$
    * Note we can obtain this by the cosine law
  - $T_2 = \frac{1}{2}m_2\left((l\dot{\theta}_1)^2 + \left(\frac{1}{2}l(\dot{\theta}_1 + \dot{\theta}_2)\right)^2 + l^2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2\right) + \frac{1}{2}I_{\otimes,2}(\dot{\theta}_1 + \dot{\theta}_2)^2$

    $= \frac{1}{2}m_2(l\dot{\theta}_1)^2 + c_2(l\dot{\theta}_1)(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2$
    * $c_2 = \frac{1}{2}m_2 l, I_2 = I_{\otimes,2} + \frac{1}{4}m_2 l^2$
    * c_2 is the first moment of mass of link 2 about its joint
  - $V_1 = \frac{1}{2}m_1 gl\sin\theta_1, V_2 = m_1 gl\left(\sin\theta_1 + \frac{1}{2}\sin(\theta_1 + \theta_2)\right)$
  - Virtual work done at joints: $\widehat{\delta W}_2 = \tau_1\delta\theta_1, \widehat{\delta W}_2 = \tau_2\delta\theta_2$
  - $(I_1 + I_2 + m_2 l^2 + 2c_2 l\cos\theta_2)\ddot{\theta}_1 + (I_2 + c_2 l\cos\theta_2)\ddot{\theta}_2 - c_2 l(2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2\sin\theta_2 + \left(\frac{1}{2}m_1 + m_2\right)gl\cos\theta_1 + \frac{1}{2}m_2 gl\cos(\theta_1 + \theta_2) = \tau_1$
  - $(I_2 + c_2 l\cos\theta_2)\ddot{\theta}_1 + I_2\ddot{\theta}_2 + c_2 l\dot{\theta}_1^2\sin\theta_2 + \frac{1}{2}m_2 gl\cos(\theta_1 + \theta_2) = \tau_2$
  - We can cast this in a general form: $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^g(\boldsymbol{q}) = \boldsymbol{u}(t)$
    * $\boldsymbol{q} = \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}$ are the coordinates
    * $\boldsymbol{u} = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}$ are the applied forces

* $\boldsymbol{f}^g(\boldsymbol{q}) = \begin{bmatrix} -\left(\frac{1}{2}m_1 + m_2\right) gl\cos\theta_1 - \frac{1}{2}m_2 gl\cos(\theta_1+\theta_2) \\ -\frac{1}{2}m_2 gl\cos(\theta_1+\theta_2) \end{bmatrix}$
* $\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} I_1 + I_2 + m_2 l^2 + 2c_2 l\cos\theta_2 & I_2 + c_2 l\cos\theta_2 \\ I_2 + c_2 l\cos\theta_2 & I_2 \end{bmatrix}$
  - This is the mass matrix, and it's symmetric positive definite
* $\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} -c_2 l(2\dot\theta_1 + \dot\theta_2)\dot\theta_2\sin\theta_2 \\ c_2 l\dot\theta_1^2\theta_2 \end{bmatrix}$

- In general, the kinetic energy of link i is $T_i = \frac{1}{2}m_i\mathbf{v}_i^T\mathbf{v}_i - \mathbf{v}_i^T\boldsymbol{c}_i^\times\boldsymbol{\omega}_i + \frac{1}{2}\boldsymbol{\omega}_i^T\boldsymbol{I}_i\boldsymbol{\omega}_i$
  - Note this uses O_i as a reference point
  - We may write this as $T_i = \frac{1}{2}\boldsymbol{v}_i^T\boldsymbol{M}_i\boldsymbol{v}_i$
  - $\boldsymbol{v}_i = \begin{bmatrix}\mathbf{v}_i \\ \boldsymbol{\omega}_i\end{bmatrix}, \boldsymbol{M}_i = \begin{bmatrix} \boldsymbol{m_i 1} & -\boldsymbol{c}_i^\times \\ \boldsymbol{c}_i^\times & \boldsymbol{I}_i\end{bmatrix}$
  - Note $\boldsymbol{M}_i = \int \begin{bmatrix}\mathbf{1} \\ \boldsymbol{s}^\times\end{bmatrix}\begin{bmatrix}\mathbf{1} \\ \boldsymbol{s}^\times\end{bmatrix}^T \mathrm{d}m$
  - These are the generalized velocity and mass matrices
- We can build a Jacobian for link i like the end-effector: $\boldsymbol{v}_i = \boldsymbol{J}_1(q_1,\dots,q_i)\dot{\boldsymbol{q}}$
  - Note that this is a function of only the coordinates up to q_i, since we have a serial manipulator
  - Therefore columns of $\boldsymbol{J}_i$ after column I are zero
  - Also, this Jacobian is expressed in the link frame, instead of the world frame
    * i.e. $\boldsymbol{J}_i = \begin{bmatrix}\boldsymbol{C}_{i,0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}_{i,0}\end{bmatrix}\boldsymbol{J}_i'$, where the latter is in the world frame
  - Note for a prismatic joint, $\boldsymbol{M}_i$ is dependent on d_i (since the reference point O_i changes with d_i)
    * With the moving reference point, $\boldsymbol{s}$ changes with d_i
    * Suppose we pick some reference point O_i' fixed to the link, with position $\boldsymbol{r}$ relative to O_i, then $\boldsymbol{s} = \boldsymbol{r} + d_i\mathbf{1}_3$
    * $\begin{bmatrix}\mathbf{1} \\ \boldsymbol{s}^\times\end{bmatrix} = \begin{bmatrix}\mathbf{1} & \mathbf{0} \\ d_i\mathbf{1}_3^\times & \mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{1} \\ \boldsymbol{r}^\times\end{bmatrix} = \boldsymbol{D}\begin{bmatrix}\mathbf{1} \\ \boldsymbol{r}_\times\end{bmatrix}$
    * Therefore $\boldsymbol{M}_i = \boldsymbol{D}\boldsymbol{M}_{i,O_i'}\boldsymbol{D}^T$, where $\boldsymbol{M}_{i,O_i'}$ is independent of d_i, but $\boldsymbol{D}$ is
    * This doesn't really matter in the end since we need to add up all the kinetic energies in the end
- $T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{J}_i^T(\boldsymbol{q})\boldsymbol{M}_i\boldsymbol{J}_i(\boldsymbol{q})\dot{\boldsymbol{q}}$
  - For the whole manipulator, $T = \sum_{i=1}^n T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T\left[\sum_{i=1}^n \boldsymbol{J}_i^T(\boldsymbol{q})\boldsymbol{M}_i\boldsymbol{J}_i(\boldsymbol{q})\right]\dot{\boldsymbol{q}}$
    * We can define the middle part as the mass matrix for the whole arm
    * $\boldsymbol{M}(\boldsymbol{q}) = \sum_{i=1}^n \boldsymbol{J}_i^T(\boldsymbol{q})\boldsymbol{M}_i\boldsymbol{J}_i(\boldsymbol{q})$
  - $T = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$
    * Note that this $\boldsymbol{M}$ is symmetric and positive definite, since if any joint is moving, we will have some amount of kinetic energy
- For gravitational potential energy, the centre of mass of link i is $\boldsymbol{r}_0^{i,\otimes} = \begin{bmatrix}\boldsymbol{\rho}_0^{i,\otimes} \\ 1\end{bmatrix} = \begin{bmatrix}\boldsymbol{C}_{0,i} & \boldsymbol{\rho}_0^i \\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{\rho}_i^{i,\otimes} \\ 1\end{bmatrix} = \boldsymbol{T}_{0,i}\boldsymbol{r}_i^{i,\otimes}$
  - The height is then $h^{i,\otimes} = \boldsymbol{k}^T\boldsymbol{r}_0^{i,\otimes} = \boldsymbol{k}^T\boldsymbol{T}_{0,i}\boldsymbol{r}_i^{i,\otimes}$
  - $\boldsymbol{k} = \begin{bmatrix}\mathbf{1}_3 \\ 0\end{bmatrix}$
- Therefore $V_i = m_i g\boldsymbol{k}^T\boldsymbol{T}_{0,i}\boldsymbol{r}_i^{i,\otimes}$ so $\boldsymbol{V} = \sum_{i=1}^n V_i = \sum_{i=1}^n m_i g\boldsymbol{k}^T\boldsymbol{T}_{0,i}\boldsymbol{r}_i^{i,\otimes}$

- The virtual work done is $\widehat{\delta W}_i^{\text{con}} = u_i \delta q_i$ so $\widehat{\delta W}^{\text{con}} = \sum_i u_i \delta q_i = \delta \boldsymbol{q}^T \boldsymbol{u}$
  - If we have friction, $\widehat{\delta W}_i^f = f_i(q_i, \dot{q}_i)\delta q_i$
  - Then $\Delta \widehat{W}^f = \sum_i f_i \delta q_i = \delta \boldsymbol{q}^T \boldsymbol{f}^f(\boldsymbol{q}, \dot{\boldsymbol{q}})$
  - If we have forces at the end-effector, we also have $\widehat{\delta W}^{ee} = \delta \boldsymbol{q}^T \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{f}^{ee}$
- The total non-conservative virtual work is then $\widehat{\delta W} = \delta \boldsymbol{q}^T(\boldsymbol{u} + \boldsymbol{f}^f + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{f}^{ee})$
- The resulting equation of motion is $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^f(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}^g(\boldsymbol{q}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{f}^{ee} = \boldsymbol{u}(t)$
  - $\boldsymbol{f}^f$ are the frictional forces, $\boldsymbol{f}^g$ are the gravitational forces and $\boldsymbol{f}^{ee}$ are the forces at the end-effector
  - $\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a nonlinear inertial term
  - $h_k = \sum_{j=1}^{n} \left(\dot{M}_{kj} - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial M_{ij}}{\partial q_k} \dot{q}_i \right) \dot{q}_j$
- Like kinematics, we have 2 problems: inverse dynamics (given a trajectory $\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}$, solve for $\boldsymbol{u}(t)$), and forward or simulation dynamics (given control forces $\boldsymbol{u}(t)$, solve for the motion $\boldsymbol{q}$)
  - Inverse dynamics is much easier than forward dynamics
  - The problem is even more complex if the manipulator links are elastic, which is an issue for space manipulators especially
  - Flexibility/compliance at the joints also complicates the problem