Lecture 20, Nov 23, 2023

Geometry in $SE(3)$

- We can represent the position of the end-effector using $SE(3)$ transformations
  - $\boldsymbol{u}^{ee} = \left(\prod_{i=1}^{n} \boldsymbol{T}_{i-1}\right) \boldsymbol{u}_n^{n+1}$ (note the $\boldsymbol{u}_n^{n+1}$ brings us from the last joint to the end-effector)
  - In matrix form: $\begin{bmatrix} \boldsymbol{r}^{ee} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_{i-1,i} & \boldsymbol{\rho}_{i-1}^{i} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho}_n^{n+1} \\ 1 \end{bmatrix}$
- The orientation of the end-effector is given by $\boldsymbol{C}^{ee} = \prod_{i=1}^{n} \boldsymbol{C}_{i-1,i}$
- We can combine both into the pose: $\boldsymbol{T}^{ee} = \prod_{i=1}^{n+1} \boldsymbol{T}_{i-1,i}$
  - $\boldsymbol{T}^{ee} = \begin{bmatrix} \boldsymbol{C}_{0,n} & \boldsymbol{r}^{ee} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$
  - $\boldsymbol{T}_{n,n+1} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{\rho}_n^{n+1} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$
    - We added this to bring us from the last joint to the end-effector

Inverse Kinematics

- Technically inverse "geometry"
- In general, $\boldsymbol{r}^{ee} = \boldsymbol{f}_r(\boldsymbol{q}), \boldsymbol{\theta}^{ee} = \boldsymbol{f}_\theta(\boldsymbol{q}) \implies \boldsymbol{p}^{ee} = \boldsymbol{f}(\boldsymbol{q})$ where $\boldsymbol{p}^{ee}$ is the end-effector pose
  - Given $\boldsymbol{q}$, solving for $\boldsymbol{p}^{ee}$ is the problem of *forward kinematics*
  - Given $\boldsymbol{p}^{ee}$, solving for $\boldsymbol{q}$ is the problem of *inverse kinematics*
- Solving inverse kinematics often requires numerical techniques, and often has multiple (possibly infinite) solutions

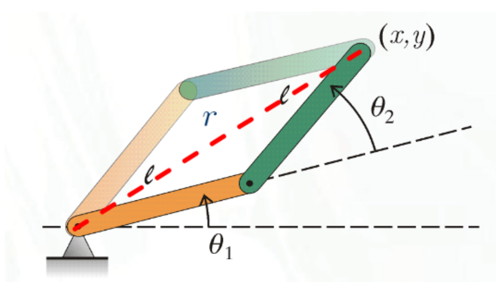

Figure 1: Example of a two-link system where multiple solutions exist.

- For the example above, we can solve for the angles using the cosine law; the $\cos^{-1}$ gives two possible solutions, one for positive θ_2 and one for negative θ_2
- A 6-DoF robotic arm with revolute joints can have as many as 16 solutions depending on the link lengths
- Incremental solution technique: given a solution at $\boldsymbol{q}$ corresponding to $\boldsymbol{p}^{ee}$, what if we changed $\boldsymbol{p}^{ee}$ by a small $\Delta\boldsymbol{p}^{ee}$?
  - $\begin{bmatrix} \boldsymbol{v}^{ee} \\ \boldsymbol{\omega}^{ee} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \implies \begin{bmatrix} \Delta t \boldsymbol{v}^{ee} \\ \Delta t \boldsymbol{\omega}^{ee} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\Delta q$
  - Therefore $\Delta\boldsymbol{p}^{ee} = \begin{bmatrix} \Delta\boldsymbol{r}^{ee} \\ \Delta\boldsymbol{\phi}^{ee} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\Delta\boldsymbol{q}$ (since for small angular displacements only, we can directly multiply by Δt to get $\Delta\boldsymbol{\phi}$)
  - Notice that this look exactly like the kinematical relation; we can now use the pseudoinverse to solve for it

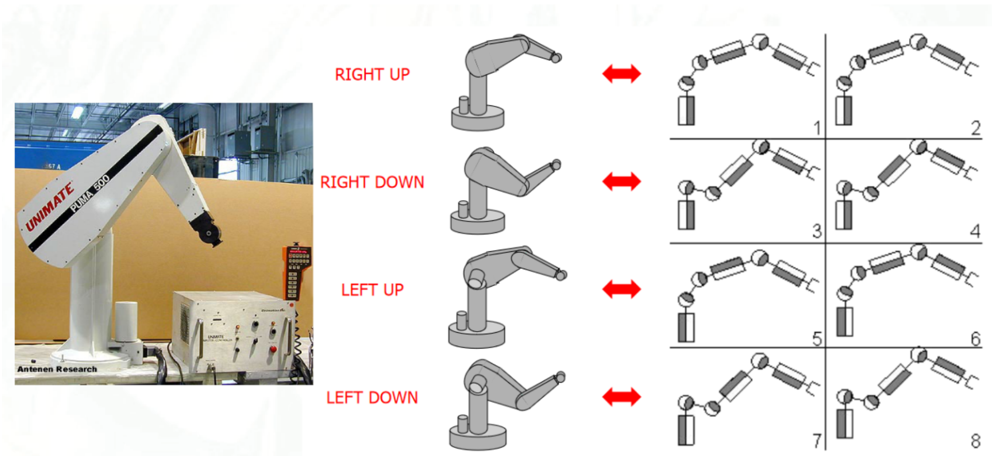

Figure 2: Example of a system with even more solutions.

- $\Delta \boldsymbol{q} = \boldsymbol{J}^\dagger(\boldsymbol{q})\Delta \boldsymbol{p}^{ee} + (\mathbf{1} - \boldsymbol{J}^\dagger \boldsymbol{J})\boldsymbol{b}$ where $\boldsymbol{J}^\dagger = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$
  - But this doesn't quite do it because the inverse can be big, even when Δp^{ee} is small
- The *damped least-squares technique* (aka Levenberg-Marquardt method) is a variation on the incremental technique
  - Minimize $\|\Delta \boldsymbol{p}^{ee} - \boldsymbol{J}(\boldsymbol{q})\Delta \boldsymbol{q}\|^2 + \lambda^2 \|\Delta \boldsymbol{q}\|^2$
  - λ is a damping term which makes sure that our Δqs are small – this is known as *regularization*
    * If we're talking about a pose, we can use $\boldsymbol{T}$ and use a matrix norm
  - This is equivalent to minimizing $\left\| \begin{bmatrix} \Delta \boldsymbol{p}^{ee} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{J} \\ \lambda \mathbf{1} \end{bmatrix} \Delta \boldsymbol{q} \right\|$, which is like a linear regression minimizing $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$
    * Therefore this is satisfied by $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b} \implies \begin{bmatrix} \boldsymbol{J} \\ \lambda \mathbf{1} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{J} \\ \lambda \mathbf{1} \end{bmatrix} \Delta \boldsymbol{q} = \begin{bmatrix} \boldsymbol{J} \\ \lambda \mathbf{1} \end{bmatrix}^T \begin{bmatrix} \Delta \boldsymbol{p}^{ee} \\ \mathbf{0} \end{bmatrix}$
  - This reduces to $\Delta \boldsymbol{q} = (\boldsymbol{J}^T\boldsymbol{J} + \lambda^2 \mathbf{1})^{-1} \boldsymbol{J}^T \Delta \boldsymbol{p}^{ee}$
    * The addition of the $\lambda \mathbf{1}$ term regularizes the solution and keeps the inverse small even when $\boldsymbol{J}^T\boldsymbol{J}$ is close to singular

Planning

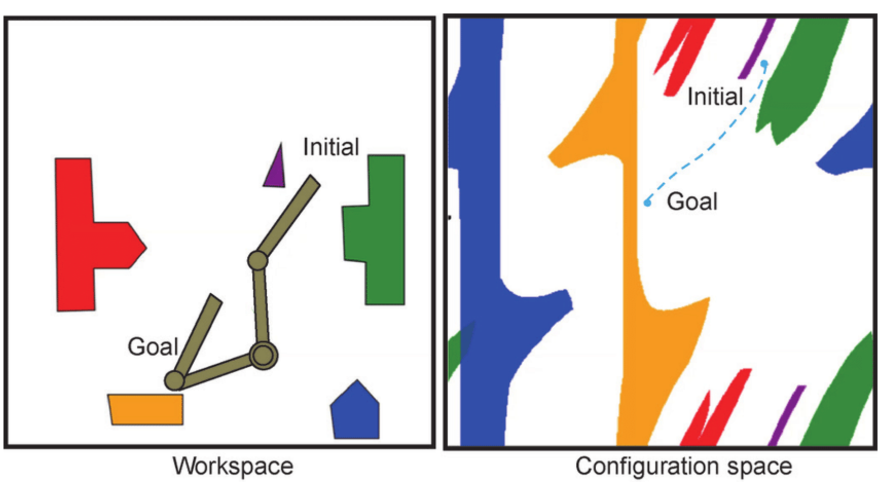

Figure 3: Mapping from workspace to configuration space.

- How do we get from work (task) space to configuration space?
- Recall that C, the configuration manifold, is the set of all possible points for the manipulator; Ω is all the parts of the configuration manifold occupied by obstacles, barriers and prohibited areas; then the free world manifold is $W = C \backslash \Omega$
- For a simple manipulator like the 2-link manipulator in 2 dimensions, we can calculate exactly where the links are and check that points on the links do not overlap obstacles
- In general, an analytical expression for this might be impossible to obtain, so we must resort to numerical methods
- The simplest way is to test point by point whether the manipulator at a given point in configuration space intersects obstacles
  - Take a point $\boldsymbol{q}$ in C and determine all the points in the manipulator, $\mathcal{M}(\boldsymbol{q})$
  - Make sure that $\mathcal{M}(\boldsymbol{q}) \cap \mathcal{O}_{i,\text{work}} = \varnothing$, then $\boldsymbol{q}$ is accessible in C
- Path planning techniques for mobile robots can also be used for manipulators in configuration space, e.g. road-map methods, Dijkstra's/A^*, potential fields, RRTs

Manipulability

- How can we measure quantitatively the ability of a manipulator to undertake a task? Can we provide a measure of the maneuverability or manipulability for a manipulator?
- We can do this kinematically or dynamically
- Consider an n-link manipulator; taking just the velocity partition, we have $\boldsymbol{v} = \boldsymbol{J}^{(v)}(\boldsymbol{q})\dot{\boldsymbol{q}}$ (we will drop the superscript from here on)
- Consider the set of all possible end-effector velocities $\boldsymbol{v}$ realizable by joint rates contained by $\|\dot{\boldsymbol{q}}\|^2 \leq 1$; intuitively, the larger this set, the more "manipulable" the manipulator is
  - Note this requires some weighting and/or non-dimensionalization if both revolute and prismatic joints are involved
  - This set will turn out to be an ellipsoid, which is called the *manipulability ellipsoid*

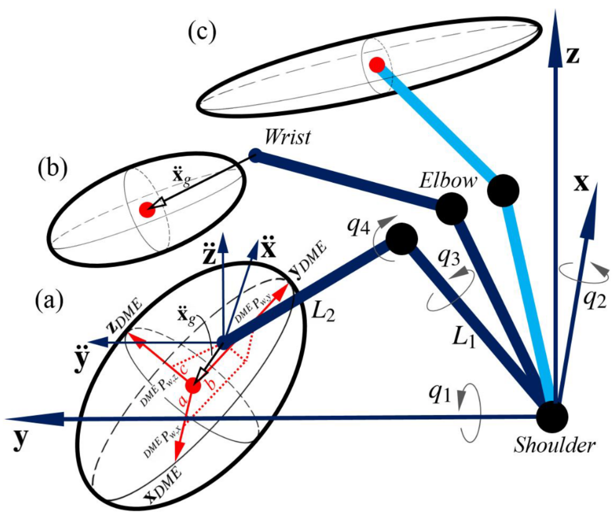

Figure 4: Manipulability ellipsoid.

- Recall that away from a singularity, $\dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger \boldsymbol{v} + (\mathbf{1} - \boldsymbol{J}^\dagger \boldsymbol{J})\boldsymbol{b}$
  - This gives $\|\dot{\boldsymbol{q}}\|^2 = \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} \geq \boldsymbol{v}^T (\boldsymbol{J}^\dagger)^T \boldsymbol{J}^\dagger \boldsymbol{v}$
  - Therefore the manipulability ellipsoid is $\boldsymbol{v}^T (\boldsymbol{J}^\dagger)^T \boldsymbol{J}^\dagger \boldsymbol{v} \leq 1$
- The principal axes of this ellipsoid represent how fast the ellipsoid can move
  - The size is given by the eigenvalues of $(\boldsymbol{J}^\dagger)^T \boldsymbol{J}^\dagger$ (like the energy/momentum ellipsoid derivation)
  - Note that substituting in the definition for $\boldsymbol{J}^\dagger$, we have $(\boldsymbol{J}^\dagger)^T \boldsymbol{J}^\dagger = (\boldsymbol{J}\boldsymbol{J}^T)^{-1}$
- Consider the SVD of $\boldsymbol{J}$: $\boldsymbol{J} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$

- For us, $m < n$ so $\boldsymbol{\Sigma}$ has several zero columns at the end
- Note that the singular values are the square roots of the eigenvalues of $\boldsymbol{J}\boldsymbol{J}^T$
- Then $\boldsymbol{J}^\dagger = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1} = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T)^{-1} = \boldsymbol{V}\boldsymbol{\Sigma}^T(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{U}^T$
- And $(\boldsymbol{J}^\dagger)^T\boldsymbol{J}^\dagger = (\boldsymbol{J}\boldsymbol{J}^T)^{-1} = \boldsymbol{U}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{U}^T$
- So $\boldsymbol{v}^T(\boldsymbol{J}^\dagger)\boldsymbol{J}^\dagger\boldsymbol{v} = \boldsymbol{v}^T\boldsymbol{U}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{U}^T\boldsymbol{v} \leq 1$ gives the ellipsoid
- Let $\boldsymbol{z} = \boldsymbol{U}^T\boldsymbol{v}$, then we have $\boldsymbol{z}^T(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{z} = 1$
- So in terms of z, we get $\frac{z_1^2}{\sigma_1^2} + \frac{z_2^2}{\sigma_2^2} + \frac{z_3^2}{\sigma_3^2} = 1$ – an ellipsoid with axes $\sigma_1, \sigma_2, \sigma_3$

- Given the ellipsoid, we can define several different measures of manipulability:
  - $w_1(\boldsymbol{q}) = \sigma_1\sigma_2\sigma_3 = \sqrt{\det(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}))}$ (ellipsoid volume)
  - $w_2(\boldsymbol{q}) = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}$ (ellipsoid stretching)
  - $w_3(\boldsymbol{q}) = \sigma_{\text{min}}$ (length of shortest axis)
  - $w_4(\boldsymbol{q}) = (\sigma_1\sigma_2\sigma_3)^{\frac{1}{3}} = w_1^{\frac{1}{3}}(\boldsymbol{q})$ (geometric mean of the axes)

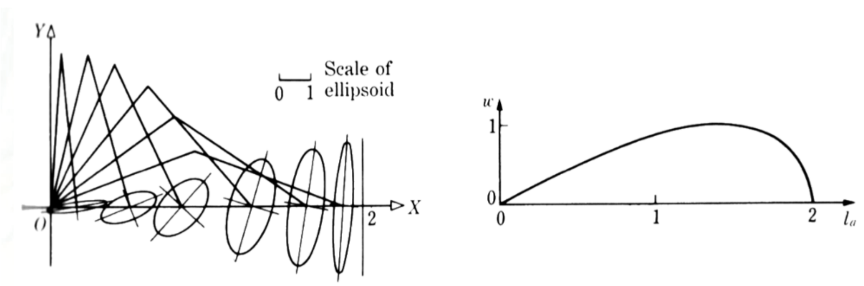

Figure 5: Manipulability ellipsiods for the 2-link manipulator.

- For the 2-link manipulator, we have $w = |\det(\boldsymbol{J})| = l^2|\sin\theta_2|$
  - Notice that this is not dependent on θ_1
  - The ellipsoid is rounder when we have intermediate values of θ_2
  - At the singularities, the ellipsoid collapses to a line, so the manipulability also drops to 0
- Recall that $\boldsymbol{\eta} = \boldsymbol{J}^T\boldsymbol{f}$, so we can address manipulability from a force/torque perspective using this dual relation
  - Consider $\|\boldsymbol{\eta}\|^2 \leq 1$
  - Going through the same steps yields $\boldsymbol{f}^T\boldsymbol{J}\boldsymbol{J}^T\boldsymbol{f} = 1 \implies \sigma_1^2 f_1^2 + \sigma_2^2 f_2^2 + \sigma_3^2 f_3^2 = 1$
  - Notice that this flips the intercepts of the ellipsoid
  - e.g. in the diagram below, larger kinematics ellipsoids result in smaller force ellipsoids

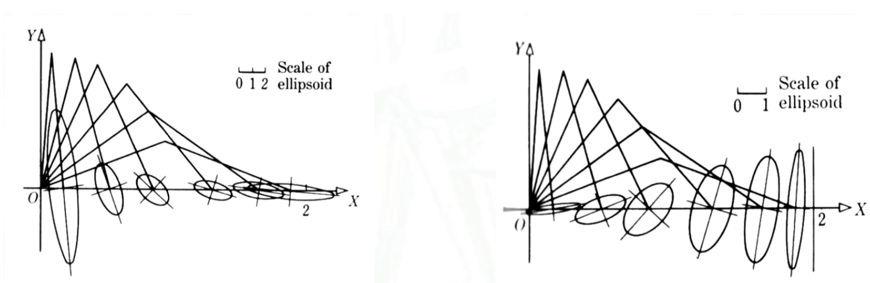

Figure 6: Force and kinematic ellipsiods for the 2-link manipulator.