Lecture 10, Oct 10, 2023

- Example: consider the problem $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$; given that constant $P \in \mathbb{R}^{n \times n}$ is positive-definite and $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}$ is negative-definite, show that $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable
  - Candidate Lyapunov function: $v(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$, which is positive-definite since $\boldsymbol{P}$ is positive-definite
  - $\dot{v}(\boldsymbol{x}) = \dfrac{\partial}{\partial t}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} = \dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}\dot{\boldsymbol{x}} = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x}$ which is negative-definite since $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}$ is negative-definite
  - Therefore by Lyapunov's method the solution $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable
  - The condition $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{L}$, where $\boldsymbol{P}, \boldsymbol{L}$ are both semi-definite, is called *Lyapunov's equation*; this condition is equivalent to saying that $\boldsymbol{A}$ has only eigenvalues with negative real parts