Lecture 2, Sep 13, 2023

Key Properties of the Laplace Transform

- The basic impulse function is given by $g(t) = \begin{cases} \lim\limits_{t_0 \to 0} \dfrac{A}{t_0} & 0 < t < t_0 \\ 0 & \text{elsewhere} \end{cases}$
  - $\mathcal{L}\{g(t)\} = A$
- Multiply by $e^{-\alpha t}$ in time domain is a shift in frequency domain
  - $\mathcal{L}\left\{e^{-\alpha t} f(t)\right\} = F(s + \alpha)$
  - By shifting the signal in the frequency domain, we introduce an exponential decay in the signal – this can be used to remove poles that we don't want to modify the steady-state behaviour
- A change of time scale in the time domain corresponds to a scaling in frequency domain
  - $\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$
  - This allows us to manipulate frequencies while keeping other important information such as phase intact
- Differentiation in time domain is equivalent to multiplication by s in the frequency domain
  - $\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right\} = sF(s) - f(0)$
  - Note that if there is an impulse at $t = 0$, two separate Laplace transforms must be used
  - This allows us to implement differentiation circuits
- Final value theorem: $\lim\limits_{t \to \infty} f(t) = \lim\limits_{s \to 0} sF(s)$
  - To find the steady-state behaviour for a stable, settling system, we can use this limit in the frequency domain, which is often much easier to evaluate
- Initial value theorem: $\lim\limits_{t \to 0^+} f(t) = \lim\limits_{s \to \infty} sF(s)$
  - This can be used to find the initial conditions, provided the limit exists in the Laplace domain
- Multiplication by t in the time domain is differentiation in frequency domain:
  - $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$
- Convolution in time domain is multiplication in Laplace domain:
  - $\mathcal{L}\{f_1(t) * f_2(t)\} = \mathcal{L}\left\{\int_0^t f_1(t - \tau) f_2(\tau) \,\mathrm{d}\tau\right\} = F_1(s) F_2(s)$
  - This can be used to easily determine the output of a system given some input signal

Circuits in Frequency Domain

- In frequency domain Ohm's law becomes $V(s) = I(s) \cdot Z$
  - The impedance Z has both real and imaginary components, unlike in normal Ohm's law
- Resistors behave the same regardless of frequency, so $Z = R$ for resistors
- However capacitors and inductors behave differently depending on frequency

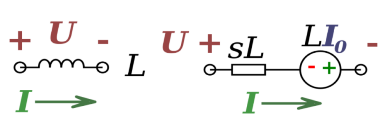

Figure 1: An inductor in time and frequency domain.

- An inductor in frequency domain is separated into two parts: the impedance $Z = sL$ for the steady-state conditions, and another part that responds to initial conditions; when we analyze steady-state behaviour, we can ignore the second part
  - $v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \implies V(s) = sLI(s) - Li(0)$
  - Notice that for DC current, $s = j\omega = 0$ so the inductor becomes an open circuit in steady state
  - For higher frequencies, s increases so the impedance also increases

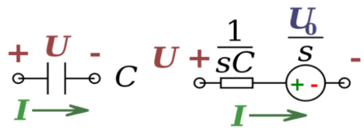

Figure 2: A capacitor in time and frequency domain.

- A capacitor similarly has impedance $Z = \frac{1}{sC}$ and another part responding to initial conditions
  - $i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \implies I(s) = CsV(s) - CV(0) \implies V(s) = \frac{1}{sC}I(s) + \frac{1}{s}V(0)$
  - For DC current, the capacitor behaves as an open circuit (infinite impedance)
  - For higher frequencies, the impedance decreases
- Any calculations that we could do with Ohm's law in time domain, we can do in frequency domain with impedances
  - This applies to techniques such as nodal analysis, mesh current, etc
  - Crucially, superposition also holds

Summary

To transform circuits into the frequency domain:

- Resistor: $Z = R$
- Inductor: $Z = sL$ plus a source with voltage $V(s) = Li(0)$ for initial conditions
- Capacitor: $Z = \frac{1}{sC}$ plus a source with voltage $V(s) = \frac{v(0)}{s}$ for initial conditions

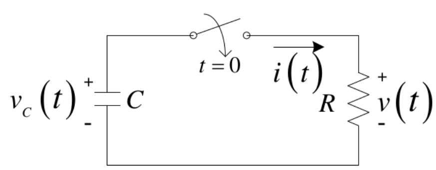

Figure 3: Example problem.

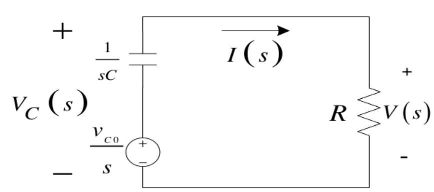

Figure 4: Example problem in frequency domain.

- Example: consider the circuit above where the capacitor is charged to a voltage of $v_c = v_{c_0}$; the switch is closed at time $t = 0$; find the resistor voltage $v(t)$
  - First transform the circuit into frequency domain as shown above; note the capacitor transforms into two sources due to initial conditions
  - $\frac{v_{c_0}}{s} - I(s)\frac{1}{sC} - I(s)R = 0$
    * Note that we're looking for $V(s) = I(s)R$
    * $I(s)\left(R + \frac{1}{sC}\right) = \frac{v_{c_0}}{s} \implies I(s) = \frac{\frac{v_{c_0}}{s}}{R + \frac{1}{sC}} = \frac{\frac{1}{R}}{s + \frac{1}{RC}}v_{c_0}$
      - Tip: to save time, avoid merging terms or normalizing until the very end

- To normalize, we should do it so that the highest power of s in the dominator has a coefficient of 1
- We can clearly see that there is a pole at $s = -\dfrac{1}{RC}$, corresponding to an exponential decay as we expected

* Multiplying by R we get $V(s) = \dfrac{1}{s + \frac{1}{RC}} v_{c_0}$ so $v(t) = u(t) v_{c_0} e^{-\frac{1}{RC}t}$