Lecture 18, Nov 15, 2023

Applications of DFS – Cycle Detection

Theorem

White-Path-Theorem (WPT): For all graphs G and all DFS of G, v becomes a descendant of u if and only if at the time $d[u]$ the DFS discovers u, there exists a path from u to v that consists of entirely white nodes.

- Proof of WPT: consider any G and any DFS of G
  - Suppose v is a descendant of u:
    * Let the discovery path from u to v be $u \to u_1 \to \cdots \to u_k \to v$
    * At the time $d[u]$ when u is discovered, all the other nodes on the path have not yet been discovered
    * Therefore all the nodes in the discovery path are white, so there exists a white path from u to v
  - Suppose at time $d[u]$ when u is discovered, there exists a white path from u to v
    * Claim: all nodes in this path, including v, will become descendants of u
    * Suppose there is at least one node along this path that does not become a descendant of u; let z be the closest node to u in the path that does not become a descendant of u
    * Let w be the node before z in this graph; since w is closer to u than z, it must be either u or a descendant of u
    * We know $d[u] < d[z]$ because z is white when u was discovered
    * We know $d[z] < f[w]$ because there is an edge from w to z
    * We know $f[w] \leq f[u]$ because w is either a descendant of u, or u itself
    * Therefore $d[u] < d[z] < f[w] \leq f[u]$; because z was discovered between the discovery and exploration of u, so $d[u] < d[z] < f[z] < f[u]$, but this means z is a descendant of u
    * This leads to a contradiction, so all nodes in the path will become descendants of u
- Now that we have proven WPT we can use it for cycle detection

Theorem

A directed graph G has a cycle if and only if any/every DFS of G has a back edge.

- Proof: consider any directed G and any DFS of G
  - Suppose the DFS of G has a back edge (v, u)
    * v is a descendant of u in the DFS, so there is a discovery path u to v
    * Hence we have a cycle by going from u to v via the discovery path, and then back to u via the back edge
  - Suppose G has a cycle C
    * Let u be the first node in C that the DFS discovers
    * Let v be the node right before u in C
    * At time $d[u]$ when the DFS discovers u, all nodes in the path from u to v in C are still white, including v (since u is the first node in the entire cycle that is discovered)
    * By WPT, v eventually becomes a descendant of u in the DFS
    * When v becomes a descendant of u, we explore it, it has an edge to u; this is a back edge, since v is a descendant of u
- After we do a DFS, how do we figure out whether an edge (u, v) is a back edge?
  - When we go down the edge, we can check that both u and v are grey
  - If we're already done the DFS, then we can check whether $d[v] < d[u] < f[u] < f[v]$