Lecture 4, Oct 4, 2023

Capital Asset Pricing Model (CAPM)

- Valuation is the analytical process of determining the current or projected worth of an asset or a company
  - Performing a valuation requires projected cash-flows and an interest rate determined by risk
  - Data from "the market" (i.e. the stock market) is used to price risk
- Financial risk is defined as the uncertainty in a future payoff
- Modern portfolio theory (MPT) provides a mathematical framework for assembling a portfolio of assets to maximize return for a given risk level
  - The risk and return characteristics of an investment should not be viewed alone, but evaluated by how it affects the overall portfolio

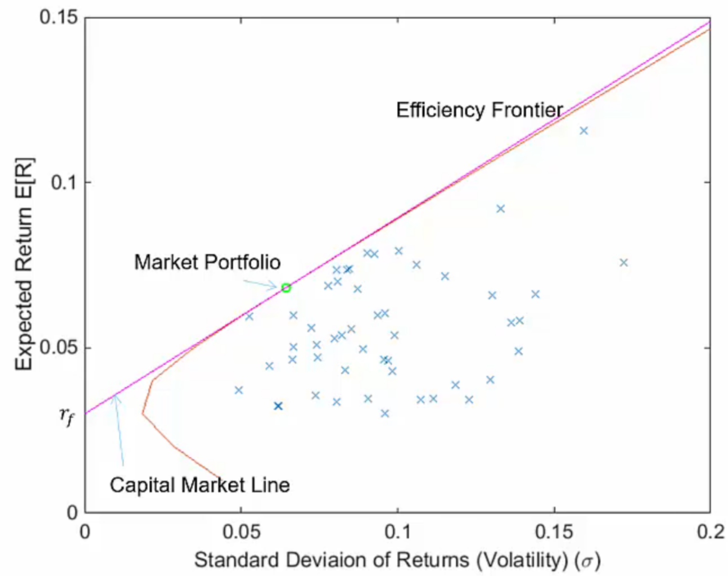

Figure 1: Expected return vs. volatility.

- Consider a portfolio of stocks; define the return vector for the ith stock as $\vec{R}_i = \begin{bmatrix} r_{t_1} \\ r_{t_2} \\ \vdots \\ r_{t_m} \end{bmatrix}$ where r_{t_j} is the (continuous) return of that stock at time t_j
  - Define the volatility σ such that $\sigma_i^2 = \text{var}(\vec{R}_i)$, which we can calculate using past historical market data
  - If we assume that we can invest some fraction x_i of our total capital in stock i, then we can try to maximize the expected return $E[\vec{R}_p]$ while trying to minimize risk σ_p
  - Optimizing this gives us an *efficiency frontier* (line in yellow); assuming that we can invest/borrow at the risk-free rate, we can draw a line from the risk-free rate that is tangent to the efficiency frontier; the tangent point is known as the *market portfolio* or *tangency portfolio*
    * The market portfolio is a theoretical portfolio that contains every single investment in existence, with each one weighted by its value; mathematically, this maximizes the return for a given volatility
    * Since the efficiency frontier is concave down, the market portfolio and the risk-free rate are the theoretically best investments given a certain level of risk
    * Thus if we assume all investors act perfectly rationally, everyone will try to invest at either the market portfolio or interest-free rate

    * Based on the proportion of investments in each of the two categories, we can calculate the overall return rate; an investor can also take on debt at the risk-free rate and invest it at the market portfolio which increases both risk and expected return, moving up the tangent line
  - This is Capital Asset Pricing Model (CAPM)
- Note that CAPM is an ideal mathematical model for something driven by investor behaviour, so it has to make some assumptions
  - Correlations between assets are fixed and constant forever (i.e. volatilities in stocks don't change and neither do their correlations)
  - All investors act perfectly rationally to maximize their economic utility
  - All investors have access to the same information at the same time
  - Investors have an accurate conception of possible returns (i.e. the probability beliefs of investors match the true distribution of returns)
  - No taxes or transaction costs
  - Investor behaviour do not influence stock prices
  - Any investor can lend and borrow an infinite amount at the risk-free interest rate
  - All securities can be divided into parcels of any size
- CAPM gives the expected return for asset i as $E[R_i] = r_f + \beta_i(E[R_{MP}] - r_f)$ where r_f is the risk-free rate and $\beta_i = \frac{\sigma_{i,MP}}{\sigma_{MP}^2} = \frac{\rho_{i,MP}\sigma_i}{\sigma_{MP}}$
  - β is a measure of risk for the company, relative to the market as a whole
    * $\beta > 1$ means the company is more risky than the market, and $0 < \beta < 1$ means the company is less risky than the market
    * $\beta \leq 0$ is almost never seen
  - Theoretically if we plotted the return against β for every single stock, it will lie on a perfect line
    * In reality if you did this it would not end well
  - $E[R_{MP}] - r_f$ is known as the *market premium* or *risk premium*, which is the difference in rate of return between the market portfolio and risk-free rate
  - We are modeling the company return as $R_{C,t} = \alpha_C + \beta_C R_{MP,t} + \varepsilon_{C,t}$
  - Volatility in $R_{MP,t}$ represents a systematic, or market risk, that is present throughout the entire market
  - Volatility in $\varepsilon_{C,t}$ represents an idiosyncratic, or firm-specific risk
  - The idea is that $E[\varepsilon_{C,t}] = 0$, so by diversifying their investments enough, a investor should be able to completely eliminate idiosyncratic risk
    * Therefore idiosyncratic risk has no inherent value, so investors should only be rewarded for market risk
- Any company can be broken down into 3 parts: assets, equity, and debt
  - The return on equity R_E comes from CAPM
  - The return on assets is a weighted sum $R_A = \frac{E}{E+D}R_E + \frac{D}{E+D}R_D$

Arbitrage

- In general arbitrage is the idea of taking advantage of inconsistencies in prices to be able to pay a lower price than what you should have
  - In financial markets arbitrage is the practice of taking advantage of a price difference between two or more markets
  - In academic use, it refers to the opportunity to achieve a risk-free gain at a rate greater than the risk-free rate
    * A fundamental assumption is that there is no true arbitrage, or the "no free lunch" assumption
  - Statistical arbitrage occurs when a gain is achieved at a higher risk than one should for a given level of risk
- Example: if you can buy a bushel of corn for $10 today and enter into a forward contract to deliver the corn for a pre-settled price of $12 per bushel one year from now, and it costs you nothing to store the corn; say you could borrow at 10%, then this is an arbitrage opportunity, because we could just borrow money to buy corn and then sell it a year later to get free money at no risk

  - However in the real world, as demand for corn increases the price will also increase
  - Assuming the forward price of \$12 does not change, the eventually the price right now would increase to \$10.91 so that we end up with no net gain
  - This is an example of a *future* or *forward contract*, which locks in a specific price for an asset; it is used by companies or investors to hedge against risks or speculate
    * A forward contract is an obligation to buy or sell at a specific price and time, and typically not traded
      - Sellers and buyers are involved in a forward transaction, and are both obligated to fulfill their end of the contract at maturity
    * Futures are similar but they are settled daily (not just at maturity), so they can be bought and sold at any time and are traded on exchanges
    * For this course we don't care about the difference
- In general to use the no-arbitrage principle to price an investment, we find two alternative investment paths and conclude that they must bring equal returns
- Example: given yields r_{t_1}, r_{t_2} at two times t_1, t_2, calculate the appropriate forward rate that an investor could lock into today from t_1 to t_2
  - Suppose we invest P_0 today, then at t_1 we have $F_1 = P_0 e^{r_{0,1} t_1}$, or at t_2 we have $F_2 = P_0 e^{r_{0,2} t_2}$
  - If we take our money from t_1 and reinvest it, we have $F_2' = F_1 e^{r_{1,2}(t_2 - t_1)}$
  - The no arbitrage principle says that $F_2 = F_2'$, otherwise there would be an arbitrage opportunity
  - Therefore we have $P_0 e^{r_{0,2} t_2} = P_0 e^{r_{0,1} t_1} e^{r_{1,2}(t_2 - t_1)} \implies r_{1,2} = \dfrac{r_{0,2} t_2 - r_{0,1} t_1}{t_2 - t_1}$
- Another strategy is to build a portfolio with the same risk as the investment, then by no-arbitrage the worth of this portfolio is the same as the original investment
- Example: a project will have a \$140 payoff if the market goes up or a \$30 payoff if the market goes down; the risk-free rate is 5%, the market portfolio is priced at \$100 and will be worth \$120 if the market goes up, or \$95 if the market goes down; what is the valuation of the project?
  - We want to replicate a portfolio so that it has the same risk/payoff as the project
  - We will buy a units of the market portfolio and b units of the risk-free investment
    * Note a and b could be negative, in which case we would be borrowing
  - In the up-market case then our payoff would be $120a + 1.05b = 140$; in the down-market case the payoff is $95a + 1.05b = 30$
    * Note that in both cases the risk-free payoff is 1.05 since regardless of market the interest is still 5%
    * Using this we can solve to get $a = 4.4, b = -369.53$
  - Therefore the project is worth $100a + 1b = 70.48$
    * Note that we did not use any probability of the market going up or down at this point
  - If the probability if the market going up is 60%, what is the expected return on the MP and project?
    * For the MP the expected payoff is $120 \cdot 0.6 + 95 \cdot 40 = 110$, so the return is $\dfrac{110}{100} - 1 = 10\%$
    * For the project it is $\dfrac{140 \cdot 0.6 + 30 \cdot 0.4}{70.48} - 1 = 36.2\%$
    * Notice that the project has much higher return, because its volatility is much greater
  - What if the expected payoff was \$140 regardless of the market?
    * This means the project is now risk-free, so we discount by the risk-free rate $\dfrac{140}{1.05} = 133$