Lecture 2, Sep 20, 2023

Cash-Flow Analysis

Cash-Flow Diagrams

- Categories of cash flows:
  - Start of project:
    * First (capital) cost: expense to build/buy/install
  - During the project:
    * Revenue cost: receipts from sales of products or services
    * Operation and maintenance (O&M) costs: expenses incurred on a regular basis (e.g. labour)
    * Overhaul: major expenditure that occurs partway through the life of an asset
  - At the end of the project:
    * Salvage value: money gained for sale or disposal of a product
    * Scrap value: value of the materials that the item is made of
    * Disposal costs: costs to dispose of the item
  - These are collectively referred to as *project life-cycle costs*
- A cash-flow diagram summarizes the timing and magnitude of cash-flows using arrows that go up/down
  - The x axis (which is discrete) shows the timing of cash flows and y axis shows their value
    * Down is usually cash outflow, up is inflow
    * Note the end of one period is the beginning of the next; a cash flow at the beginning of year 1 would be at time 0 (now)
      - Cash flows that occur during a period are assumed to occur at the end of the period
      - Interest is compounded once per period unless specified
  - Cash flows at the same time are summed

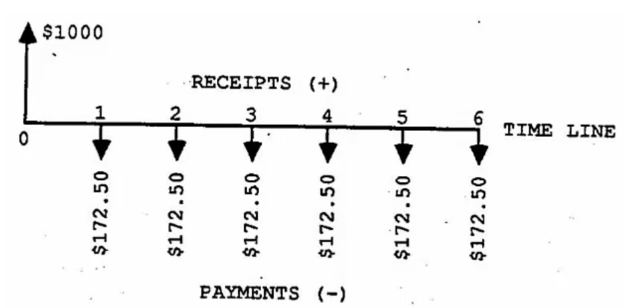

Figure 1: Example cash-flow diagram: Borrowing \$1000 from a bank and paying it back in 6 monthly installments of \$172.50 (1% interest per month)

- Cash-flows can be equivalent:
  - Mathematical equivalence: equivalence of cash-flows due to mathematical equivalence of time and money
    * Two cash-flows, P_t at time t and F_{t+N} at time $t + N$ are mathematically equivalent with respect to the interest rate i if $F_{t+N} = P_t(1 + i)^N$
  - Market equivalence: a consequence of the ability to exchange one cash-flow for another at zero cost
    * Once can exchange cash-flows between present and future amounts; *borrowing* to exchange a future cash-flow for a present one, and *lending/investing* to give up a present cash-flow for a future one
    * Example: if a T-bill of \$1000 due to mature in 6 months is selling for \$990.10, then the "market" has agreed that the appropriate interest rate is 1% over 6 months
    * The cash flow of \$1000 6 months in the future is equivalent to a \$990.10 right now
  - Decisional equivalence: due to indifferences on the part of decision maker among available choices
    * For a decision maker, two cash-flows P_t, F_{t+N} are equivalent if they are indifferent about the two

    * The interest rate is not a priori information but rather implied (which can be calculated)
  - The market tells us what interest rate we should take based on the risk; mathematical equivalence allows us to use that interest rate to do the math; decisional equivalence allows us to use the math to decide what to take

Present Value of Cash-Flows

- There are 5 common types of regular cash flows:
  - Single payments (aka receipts): a single cash payment occurring at some time
  - Perpetuity: a cash flow of magnitude A occurring at regular intervals indefinitely
  - Annuity: like a perpetuity but only over N periods
  - Arithmetic gradient: like an annuity, but the amount increases by G each period (A the first time, $A + G$ the second time, $A + 2G$ the third time and so on)
  - Geometric gradient: like the arithmetic gradient but the amount grows by a factor of $1 + g$ each period
- Factor notation: to convert from cash flow Y to X, use the factor $X = Y(X/Y, i, N)$ where i is the interest rate and N is the time period
  - X and Y are P (present), F (future), A (annuity), G (arithmetic gradient) or Geom (geometric gradient)
  - For the geometric gradient we also need the growth rate, so $X = G(X/G, i, g, N)$
  - Some factors have special names:
    * F/P: compound amount factor
    * P/F: present worth factor
    * A/F: sinking fund factor
    * F/A: series compound amount factor
    * A/P: capital recovery factor
    * P/A: series present worth factor
  - Reversing X and Y simply inverts the factor
- P/F, F/P factors can be derived by simply applying interest: $(F/P, i, N) = (1 + i)^N$
- $(P/A, i, N) = \frac{1}{i} - \frac{1}{i(1+i)^N}$
  - This can be derived through a series sum or by noting that a perpetuity of amount A has present value $\frac{A}{i}$, since the amount of interest per period is $A = Pi$
- For an arithmetic gradient, $P = A(P/A, i, N) + G(P/G, i, N)$ where $(P/G, i, N) = \frac{1}{i^2}\left(1 - \frac{1 + iN}{(1+i)^N}\right)$
  - The first part, $A(P/A, i, N)$ is the value of the initial base annuity
  - The second part is the value of the growth
  - This can be derived by turning the arithmetic gradient into a series of annuities
  - $P = G\sum_{k=1}^{N-1} \frac{(P/A, i, N-k)}{(1+i)^k}$ ignoring the annuity starting on year 1
- $(P/Geom, i, g, N) = \frac{1}{1+g}(P/A, i^0, N)$ where $i^0 = \frac{1+i}{i-g} - 1$
  - Equivalently $(P/Geom, i, g, N) = \frac{1 - \left(\frac{1+g}{1+i}\right)^N}{i - g}$
  - Note for a geometric gradient cash flow, there is nothing at time 0, then at time 1 a cash flow of A, then at time 2 a cash flow of $A(1 + g)$ and so on

Examples

- Consider annuity that pays \$10 per month for 1 year. If interest is 1% per month compounded monthly, what is the equivalent continuous payment right now?
  - $P = 10(P/A, 1\%, 12) = \$112.55$

  - $P = \int_0^1 Ae^{-r_{cc}t}\,\mathrm{d}t$ and we can find $(1+1\%)^{12} = e^{r_{cc}} \implies r_{cc} = 11.94\%/\mathrm{yr}$
  - $\$112.55 = A\int_0^1 e^{-0.1194t}\,\mathrm{d}t \implies A = \$119.40\mathrm{yr}^{-1}$
- Determine the effective annual rate for an investment that earns 15% per year based on quarterly compounding for the first 4 months, then 11% per year based on continuous compounding for another 5 months
  - $\left(1+\frac{15\%}{4}\right)^{\frac{4}{3}} e^{11\%\cdot\frac{5}{12}} = (1+r_{eff})^{\frac{9}{12}}$
- A property was sold for \$10M. The buyer wants a VTB (vendor-take-back) mortgage for \$5M, with 4% interest and \$500k of principal paid yearly for 10 years; what is the present value to the seller at 10% interest per year?
  - VTB is if the buyer defaults, the seller gets the building back
  - The buyer first gets \$5M in year 0, then \$500k each year plus some interest each year
    * As the year goes on the interest decreases, because the principal that the interest applies to decreases
    * In year 1 the interest is $\$5000000 \cdot 4\%$, in year 2 it's $\$4500000 \cdot 4\%$ and so on
    * This is an arithmetic gradient with a negative gradient
  - We have $A = 4\% \times \$5000000 = \$200000, G = -\$500000 \cdot 4\% = \$20000$
  - $P = 5000000 + (500000 + 200000)(P/A, 10\%, 10) - 20000(P/G, 10\%, 10) = 8843370$
    * Note we had to add 200k to the annuity value because of the declining interest