Lecture 9, Oct 5, 2023

A Grammar (System) of Particles

- Consider a system of particles, $\mathcal{P}_i, \mathcal{P}_j, \ldots$, with the reference points $O_{\mathfrak{I}}$, an inertially fixed reference point, $\odot$, the centre-of-mass reference point, and O, an arbitrary reference point
  - Each particle has positions $\underset{\to}{r}_i, \underset{\to}{r}_j$ relative to $O_{\mathfrak{I}}$, $\underset{\to}{s}_i, \underset{\to}{s}_j$ relative to $\odot$, masses m_i, m_j, external forces $\underset{\to}{f}_{i,ext}, \underset{\to}{f}_{j,ext}$ and forces between particles $\underset{\to}{f}_i^j, \underset{\to}{f}_j^i$
- We wish to extend the concepts of momentum and angular momentum to this grammar of particles
  - $\underset{\to}{p}_i = m_i \dot{\underset{\to}{r}}_i = m_i \underset{\to}{v}_i$
  - The total momentum is $\underset{\to}{p} = \sum_i \underset{\to}{p}_i = \sum_i m_i \dot{\underset{\to}{r}}_i$
  - $\underset{\to}{h}_i = m_i \underset{\to}{r}_i \times \dot{\underset{\to}{r}}_i = m_i \underset{\to}{r}_i \times \underset{\to}{v}_i = \underset{\to}{r}_i \times \underset{\to}{p}_i$
  - Total angular momentum is $\underset{\to}{h} = \sum_i \underset{\to}{h}_i = \sum_i m_i \underset{\to}{r}_i \times \dot{\underset{\to}{r}}_i$
- The centre of mass is $\underset{\to}{r}_{\odot} = \sum_i \frac{m_i \underset{\to}{r}_i}{m}$ where $m = \sum_i m_i$ is the total mass
  - Therefore $m\underset{\to}{r}_{\odot} = \sum_i m_i \underset{\to}{r}_i \implies m\dot{\underset{\to}{r}}_{\odot} = \sum_i m_i \dot{\underset{\to}{r}}_i = \underset{\to}{p}$ which is the total momentum
  - We can therefore work with momentum as though all particles are concentrated at the center of mass
- $\underset{\to}{r}_i = \underset{\to}{r}_{\odot} + \underset{\to}{s}_i \implies \underset{\to}{p} = \sum_i m_i(\dot{\underset{\to}{r}}_{\odot} + \dot{\underset{\to}{s}}_i) = \sum_i m_i \dot{\underset{\to}{r}}_{\odot} + \sum_i m_i \dot{\underset{\to}{s}}_i = m\dot{\underset{\to}{r}}_{\odot} + \sum_i m_i \dot{\underset{\to}{s}}_i$
  - Therefore $\sum_i m_i \dot{\underset{\to}{s}}_i = \underset{\to}{0}$; for an observer at $\odot$, the total momentum is zero
- What about forces?
  - $m_i \ddot{\underset{\to}{r}}_i = \underset{\to}{f}_{i,ext} + \sum_j \underset{\to}{f}_i^j$ assuming that $\underset{\to}{f}_i^i = \underset{\to}{0}$
  - Summing over all forces, $\sum_i m_i \ddot{\underset{\to}{r}}_i = m\ddot{\underset{\to}{r}}_{\odot} = \sum_i \underset{\to}{f}_{i,ext} + \sum_i \sum_j \underset{\to}{f}_i^j$
    * The double sum becomes $\frac{1}{2}\sum_i \sum_j (\underset{\to}{f}_i^j + \underset{\to}{f}_i^j) = \frac{1}{2}\sum_i \sum_j (\underset{\to}{f}_i^j - \underset{\to}{f}_j^i)$ by Newton's third law
    * But since we are summing over all i and j, $\frac{1}{2}\sum_i \sum_j (\underset{\to}{f}_i^j - \underset{\to}{f}_j^i) = \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_i^j - \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_j^i = \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_i^j - \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_i^j = \underset{\to}{0}$
  - Therefore $m\ddot{\underset{\to}{r}}_{\odot} = \underset{\to}{f}$, which is Newton's second law on the center of mass – we've extended the law from individual particles to systems of particles
- What about the rotational equations of motion?
  - $\dot{\underset{\to}{h}}_i = m_i \dot{\underset{\to}{r}}_i \times \dot{\underset{\to}{r}}_i + m_i \underset{\to}{r}_i \times \ddot{\underset{\to}{r}}_i = m_i \underset{\to}{r}_i \times \ddot{\underset{\to}{r}}_i = \underset{\to}{r}_i \times \underset{\to}{f}_i = \underset{\to}{r}_i \times \left(\underset{\to}{f}_{i,ext} + \sum_j f_i^j\right)$
  - So $\dot{\underset{\to}{h}} = \sum_i \dot{\underset{\to}{h}}_i = \sum_i \underset{\to}{r}_i \times \underset{\to}{f}_{i,ext} + \sum_i \sum_j \underset{\to}{r}_i \times \underset{\to}{f}_i^j = \underset{\to}{\tau} + \sum_i \sum_j \underset{\to}{r}_i \times \underset{\to}{f}_i^j$

- $\sum_i \sum_j \vec{r}_i \times \vec{f}_i^j = \frac{1}{2} \sum_i \sum_j (\vec{r}_i \times \vec{f}_i^j + \vec{r}_i \times \vec{f}_i^j)$
  $= \frac{1}{2} \sum_i \sum_j (\vec{r}_i \times \vec{f}_i^j - \vec{r}_i \times \vec{f}_j^i)$
  $= \frac{1}{2} \sum_i \sum_j (\vec{r}_i \times f_i^j - \vec{r}_j \times \vec{f}_i^j)$
  $= \frac{1}{2} \sum_i \sum_j (\vec{r}_i - \vec{r}_j) \times \vec{f}_i^j$
    * If we assume that $(\vec{r}_i - \vec{r}_j) \parallel \vec{f}_i^j$, that is, the inter-particle forces act along the lines connecting two particles, then we can make this term zero
    * This additional assumption, that equal and opposite forces act along the line connecting two particles, is the *strong form* of Newton's third law
    * This is a reasonable assumption to make because otherwise two particles in a system would keep accelerating forever
  - Hence $\dot{\vec{h}} = \vec{\tau} = \sum_i \vec{r}_i \times \vec{f}_{i,ext}$
  - Note this is not $\vec{h}_{\otimes}$, but angular momentum about the inertial reference point $O_{\mathfrak{I}}$
- $\vec{h}_{\otimes} = \sum_i s_i \times m_i(\dot{\vec{r}}_{\otimes} + \dot{\vec{s}}_i) = \sum_i m_i \vec{s}_i \times \dot{\vec{s}}_i + \left(\sum_i m_i \vec{s}_i\right) \times \dot{\vec{r}}_{\otimes} = \sum_i m_i \vec{s}_i \times \dot{\vec{s}}_i$
  - $\dot{\vec{h}}_{\otimes} = \sum_i \dot{\vec{s}}_i \times \vec{p}_i + \sum_i \vec{s}_i \times \dot{\vec{p}}_i = \sum_i \vec{s}_i \times \vec{f}_{i,ext} = \vec{\tau}_{\otimes}$
- Therefore $\dot{\vec{h}} = \vec{\tau}$ about $O_{\mathfrak{I}}$ and $\dot{\vec{h}}_{\otimes} = \vec{\tau}_{\otimes}$ about $\otimes$
  - This is a special result that only holds for the center of mass!
- In general $\dot{\vec{h}}_O + \vec{v}_O \times \vec{p} = \vec{\tau}_O$ for a general O moving at $\vec{v}_O = \dot{\vec{r}}^{OO_{\mathfrak{I}}}$ with respect to $O_{\mathfrak{I}}$
  - If $\vec{\rho}_i$ is the position of each particle about a moving reference point O, then each particle has inertial velocity $\vec{v} = \vec{v}_O + \dot{\vec{\rho}}$ so $\dot{\vec{\rho}} = \vec{v} - \vec{v}_O$
  - $\dot{\vec{h}} = \sum_i (\vec{\rho}_i \times \vec{p}_i)^{\cdot}$
    $= \sum_i m_i \dot{\vec{\rho}}_i \times \vec{v}_i + \sum_i m_i \vec{\rho}_i \times \dot{\vec{v}}_i$
    $= \sum_i m_i (\vec{v}_i - \vec{v}_O) \times \vec{v}_i + \sum_i \vec{\rho}_i \times \vec{f}_i$
    $= -\vec{v}_O \times \sum_i m_i v_i + \sum_i \vec{\tau}_i$
    $= -\vec{v}_O \times \vec{p} + \vec{\tau}_O$
  - In the inertially fixed point, $\vec{v}_O = 0$
  - At the center of mass, $\vec{p} = m\dot{\vec{r}}_{\otimes} = m\vec{v}_O$, so when we cross it with $\vec{v}_O$, the term cancels
  - In both special frames we do not need to apply a correction
- What about work and kinetic energy?
  - $W_i = \int_A^B \vec{f}_i \cdot \mathrm{d}\vec{r}_i$
  - $W = \sum_i W_i = \sum_i T_i^B - \sum_i T_i^A = T_B - T_A$ by the principle of work and kinetic energy
  - $T_i = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$ and $\dot{\vec{r}}_i = \dot{\vec{r}}_{\otimes} + \dot{\vec{s}}_i$, so substituting in:

- $T = \sum_i T_i$

  $= \frac{1}{2}\sum_i m_i(\dot{\underset{\to}{r}}_{\odot} \cdot \dot{\underset{\to}{r}}_{\odot} + 2\dot{\underset{\to}{s}}_i \cdot \dot{\underset{\to}{r}}_{\odot} + \dot{\underset{\to}{s}}_i \cdot \dot{\underset{\to}{s}}_i)$

  $= \frac{1}{2}mv_{\odot}^2 + \left(\sum_i m_i \dot{\underset{\to}{s}}_i\right) \cdot \underset{\to}{v}_{\odot} + \frac{1}{2}\sum_i m_i \dot{\underset{\to}{s}}_i \cdot \dot{\underset{\to}{s}}_i$

  $= \frac{1}{2}mv_{\odot}^2 + \frac{1}{2}\sum_i m_i u_i^2$
- So the total kinetic energy is the kinetic energy as if all the mass is at the center of mass, plus the kinetic energy of all the particles relative to the center of mass
- If we have a rigid body in translation, all $u_i = 0$, so the kinetic energy is just the same as if the mass is concentrated at the center of mass
  * Note this does not apply in rotation – in that case the second term would result in rotational kinetic energy

Summary

For a grammar of particles, the total momentum and angular momentum are defined as:

$$\underset{\to}{p} = \sum_i \underset{\to}{p}_i = \sum_i m_i \dot{\underset{\to}{r}}_i \qquad \underset{\to}{h} = \sum_i \underset{\to}{h}_i = \sum_i m_i \underset{\to}{r}_i \times \dot{\underset{\to}{r}}_i$$

The centre of mass, located at $\underset{\to}{r}_{\odot} = \frac{\sum_i m_i \underset{\to}{r}_i}{\sum_i m_i}$, satisfies:

$$m\underset{\to}{r}_{\odot} = \underset{\to}{p} \qquad m\ddot{\underset{\to}{r}}_{\odot} = \underset{\to}{f} = \sum_i \underset{\to}{f}_{i,ext}$$

The total angular momentum about O, moving at $\underset{\to}{v}_O$ relative to inertial space, in general satisfies:

$$\dot{\underset{\to}{h}}_O + \underset{\to}{v}_O \times \underset{\to}{p} = \underset{\to}{\tau}_O = \sum_i \underset{\to}{\rho}_i \times \underset{\to}{f}_{i,ext}$$

When $O = \odot$ or inertially fixed, this reduces to $\dot{\underset{\to}{h}} = \underset{\to}{\tau}$.

D'Alembert's Principle

- Let $\underset{\to}{\rho}_i$ be the position of particle i with respect to O, an arbitrary, possibly accelerating reference point, so $\underset{\to}{r}_i = \underset{\to}{r}^{OO_\mathfrak{I}} + \underset{\to}{\rho}_i$
- $m\ddot{\underset{\to}{\rho}}_{\odot} = \underset{\to}{f} \implies m(\ddot{\underset{\to}{r}}^{OO_\mathfrak{I}} + \ddot{\underset{\to}{\rho}}_{\odot}) = \underset{\to}{f}$
- Let $\ddot{\underset{\to}{r}}^{OO_\mathfrak{I}} = \underset{\to}{a}_O$ be the acceleration of O with respect to $O_\mathfrak{I}$; then $m\ddot{\underset{\to}{\rho}}_{\odot} = \underset{\to}{f} - m\underset{\to}{a}_O$
  - Note we've made Newton's second law work by applying a "reversed inertial force"
  - This is the essence of d'Alembert's Principle – the ability to transform ourselves into a noninertial frame but still use Newton's second law