Lecture 8, Oct 3, 2023

Rotational Version of Newton's Laws

- Recall the momentum is $\underset{\rightarrow}{p} = m\dot{\underset{\rightarrow}{r}}$
- $\underset{\rightarrow}{h}_O = \underset{\rightarrow}{r} \times \underset{\rightarrow}{p}$ is the *angular momentum*, or moment of momentum
- $\dot{\underset{\rightarrow}{h}}_O = \dot{\underset{\rightarrow}{r}} \times \underset{\rightarrow}{p} + \underset{\rightarrow}{r} \times \dot{\underset{\rightarrow}{p}} = m\dot{\underset{\rightarrow}{r}} \times \dot{\underset{\rightarrow}{r}} + \underset{\rightarrow}{r} \times \underset{\rightarrow}{f} = \underset{\rightarrow}{r} \times \underset{\rightarrow}{f} = \underset{\rightarrow}{\tau}_O$ is the *torque*, or moment of force
  - Note that when we talk about moments such as angular momentum or torque, we need some reference point O
  - Here O is assumed to be inertially fixed; if it moves, then $\underset{\rightarrow}{\tau}_O = \dot{\underset{\rightarrow}{h}}_O + \underset{\rightarrow}{v}_O \times \underset{\rightarrow}{p}$ where $\underset{\rightarrow}{v}_O$ is the moment with respect to inertial space
- $\underset{\rightarrow}{f} = \dot{\underset{\rightarrow}{p}} = \mathring{\underset{\rightarrow}{p}} + \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{p}$
  - We can think of this as the *translational equation of motion*
- $\underset{\rightarrow}{\tau}_O = \dot{\underset{\rightarrow}{h}}_O = \mathring{\underset{\rightarrow}{h}}_O + \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{h}_O$
  - We can think of this as the *rotational equation of motion*
  - But this is not a law because it is derivable from the other laws and assumptions
- Impulse is defined as $\underset{\rightarrow}{\imath} = \displaystyle\int_{t_a}^{t_b} \underset{\rightarrow}{f}\,\mathrm{d}t = \underset{\rightarrow}{p}_b - \underset{\rightarrow}{p}_a$
- Rotational impulse is $\underset{\rightarrow}{\jmath}_O = \displaystyle\int_{t_a}^{t_b} \underset{\rightarrow}{\tau}_O\,\mathrm{d}t = \underset{\rightarrow}{h}_{O,b} - \underset{\rightarrow}{h}_{O,a}$

Work and Energy

- $$\begin{aligned} W &= \int_A^B \underset{\rightarrow}{f} \cdot \mathrm{d}\underset{\rightarrow}{r} \\ &= \int_A^B m\ddot{\underset{\rightarrow}{r}} \cdot \mathrm{d}\underset{\rightarrow}{r} \\ &= m\int \frac{1}{2}\mathrm{d}v^2 \\ &= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \\ &= T_B - T_A \end{aligned}$$
  - T_A, T_B are the *kinetic energy*; this is known as the principle of work and kinetic energy
  - Note $v^2 = \dot{\underset{\rightarrow}{r}} \cdot \dot{\underset{\rightarrow}{r}}$ so $\dfrac{\mathrm{d}v^2}{\mathrm{d}t} = \dfrac{\mathrm{d}\dot{\underset{\rightarrow}{r}} \cdot \dot{\underset{\rightarrow}{r}}}{\mathrm{d}t} = 2\ddot{\underset{\rightarrow}{r}} \cdot \dot{\underset{\rightarrow}{r}} = 2\ddot{\underset{\rightarrow}{r}} \cdot \dfrac{\mathrm{d}\underset{\rightarrow}{r}}{\mathrm{d}t}$
  - Therefore $\mathrm{d}v^2 = 2\ddot{\underset{\rightarrow}{r}} \cdot \mathrm{d}\underset{\rightarrow}{r}$
- A force $\underset{\rightarrow}{f}$ is *conservative* iff $\displaystyle\int_{P_a} \underset{\rightarrow}{f} \cdot \mathrm{d}\underset{\rightarrow}{r} = \int_{P_b} \underset{\rightarrow}{f} \cdot \mathrm{d}\underset{\rightarrow}{r}$ for any two paths P_a, P_b that have the same start and end points
  - Equivalently, $\underset{\rightarrow}{\nabla} \times \underset{\rightarrow}{f} = \underset{\rightarrow}{0}$ (no curl) or $\underset{\rightarrow}{f} = -\underset{\rightarrow}{\nabla} V$ or $\displaystyle\oint \underset{\rightarrow}{f} \cdot \mathrm{d}\underset{\rightarrow}{r} = 0$
  - If $\underset{\rightarrow}{f} = -\underset{\rightarrow}{\nabla} V$, then $\underset{\rightarrow}{f} \cdot \mathrm{d}\underset{\rightarrow}{r} = -\underset{\rightarrow}{\nabla} V \cdot \mathrm{d}\underset{\rightarrow}{r} = -\dfrac{\partial V}{\partial x_1}\mathrm{d}x_1 - \dfrac{\partial V}{\partial x_2}\mathrm{d}x_2 - \dfrac{\partial V}{\partial x_3}\mathrm{d}x_3 = \mathrm{d}V$
  - $\displaystyle\int_A^B \underset{\rightarrow}{f} \cdot \mathrm{d}\underset{\rightarrow}{r} = -\int \mathrm{d}V = V_A - V_B$, regardless of the path taken from A to B
- If we combine the above with the principle of work and kinetic energy, we see $V_A - V_B = T_B - T_A \implies T_A + V_A = T_B + V_B$
  - This is the *conservation of (total) energy* – under a conservative force field $\underset{\rightarrow}{f}$, the sum of kinetic and potential energies, $T + V$, is conserved
    * V is the *potential energy*
    * $T + V = E$ is the *total (mechanical) energy*