Lecture 7, Sep 28, 2023

Newton's Second Law in Noninertial Frames

- Kinematics was the study of the geometry of motion without regard for the laws of nature; now we move on to dynamics, where we attempt to describe the laws of nature
- We know that the law of inertia does not hold in an accelerating or rotating reference frame; what about Newton's second law?
- $\underrightarrow{f} = m\underrightarrow{a} = m\underrightarrow{r}^{\cdot\cdot}$, with the derivative taken with respect to some inertial frame $\underrightarrow{\mathcal{F}}_I$
  - In another frame $\underrightarrow{\mathcal{F}}_b$, $\underrightarrow{f} = mv^{\cdot}$

$$\begin{aligned} &= m(\underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{bI} \times \underrightarrow{v}) \\ &= m\underrightarrow{r}^{\cdot\cdot} \\ &= m(\underrightarrow{r}^{\circ\circ} + 2\underrightarrow{\omega}^{bI} \times \underrightarrow{r}^{\circ} + \underrightarrow{\omega}^{bI\circ} \times \underrightarrow{r} + \underrightarrow{\omega}^{bI} \times (\underrightarrow{\omega}^{bI} \times \underrightarrow{r})) \end{aligned}$$

  - So in $\underrightarrow{\mathcal{F}}_b$, $m\underrightarrow{r}^{\circ\circ} = \underrightarrow{f} - m\left(2\underrightarrow{\omega}^{bI} \times \underrightarrow{r}^{\circ} - \underrightarrow{\omega}^{bI\circ} \times \underrightarrow{r} + \underrightarrow{\omega}^{bI} \times (\underrightarrow{\omega}^{bI} \times \underrightarrow{r})\right)$
  - We can see this broken down into the Coriolis, tangential, and centrifugal forces

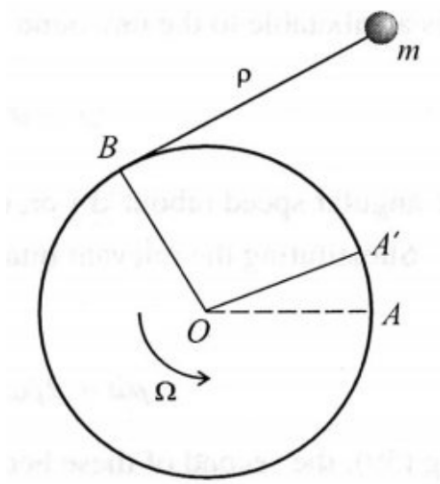

Figure 1: Diagram for the example problem.

- Example: Consider a spool with a bob attached to the end of the wire; if the spool is rotating in the opposite direction that the wire is being wound, the spool will actually unwind
  - Given that Ω is constant, what is $\rho(t)$ and $f_T(t)$?
  - Define our reference frames as $\underrightarrow{\mathcal{F}}_I$, the inertial frame, and $\underrightarrow{\mathcal{F}}_b$, a rotation reference frame with $\underrightarrow{b}_1$ parallel to the string at all times
  - This gives $\underrightarrow{r} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} \rho \\ a \\ 0 \end{bmatrix} = \underrightarrow{\mathcal{F}}_b^T \boldsymbol{r}_b$ and $\underrightarrow{f}_T = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} -f_T \\ 0 \\ 0 \end{bmatrix} = \underrightarrow{\mathcal{F}}_b^T \boldsymbol{f}_b$
  - Since $\underrightarrow{\mathcal{F}}_b$ is not an inertial frame, we must use the equation of motion for a rotating frame that we derived above
  - $\underrightarrow{f}_T = m(\underrightarrow{r}^{\circ\circ} + 2\underrightarrow{\omega}^{bI} \times \underrightarrow{r}^{\circ} + \underrightarrow{\omega}^{bI\circ} \times \underrightarrow{r} + \underrightarrow{\omega}^{bI} \times (\underrightarrow{\omega}^{bI} \times \underrightarrow{r}))$
  - It is most convenient to express all quantities in frame $\underrightarrow{\mathcal{F}}_b$:
    - $\underrightarrow{\omega}^{bI} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$
      - $\theta = \angle AOB - \dfrac{\pi}{2} = \angle AOA' + \angle A'OB - \dfrac{\pi}{2} = \Omega t + \dfrac{\rho}{a} - \dfrac{\pi}{2}$
        - Note the $\dfrac{\rho}{a}$ term comes from the fact that the arc length from B to A' is ρ

- $\dot{\theta} = \Omega + \frac{\dot{\rho}}{a}$

  * $\dot{\boldsymbol{r}}_b = \begin{bmatrix} \dot{\rho} \\ 0 \\ 0 \end{bmatrix}, \ddot{\boldsymbol{r}}_b = \begin{bmatrix} \ddot{\rho} \\ 0 \\ 0 \end{bmatrix}$

– If we substitute these quantities back in, we get $\begin{cases} -m\ddot{\rho} - m\rho\omega^2 = f_T \\ \rho\dot{\omega} + 2\dot{\rho}\omega - a\omega^2 = 0 \end{cases}$ where $\omega = \Omega + \frac{\dot{\rho}}{a}$

  * Solving the DE in the second equation, we get $\rho(t) = a\Omega t$
  * Substitute back in to get $f_T = 4ma\Omega^3 t$

– The idealized math says that the spool will keep unwinding, however in reality drag will eventually match the centrifugal force, causing the spool to no longer unwind