Lecture 5, Sep 21, 2023

Transport Equations

- We want to know how $\underrightarrow{v}^{\bullet}$ is related to $\underrightarrow{v}^{\circ}$
  - $\underrightarrow{v} = \underrightarrow{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_a = \underrightarrow{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b$
  - $\underrightarrow{v}^{\bullet} = \underrightarrow{\boldsymbol{\mathcal{F}}}_a^{T\bullet} \boldsymbol{v}_a + \underrightarrow{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_a^{\bullet} = \underrightarrow{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_b^{\bullet}$
  - $\underrightarrow{v}^{\circ} = \underrightarrow{\boldsymbol{\mathcal{F}}}_b^{T\circ} \boldsymbol{v}_b + \underrightarrow{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b^{\circ} = \underrightarrow{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b^{\circ}$
  - $\underrightarrow{v}^{\bullet} = \underrightarrow{\boldsymbol{\mathcal{F}}}_b^T \dot{\boldsymbol{v}}_b + \underrightarrow{\boldsymbol{\mathcal{F}}}_b^{T\bullet} \boldsymbol{v}_b = \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b = \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}$
    * Note that when we have as scalar $\boldsymbol{v}_b$, we put the dot over instead of to the right, because for the scalar reference frames does not matter
- Therefore $\underrightarrow{v}^{\bullet} = \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}$, which is known as the *transport equation* for velocity
- For acceleration, $\underrightarrow{v}^{\bullet\bullet} = (\underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v})^{\bullet}$

$$\begin{aligned}
&= (\underrightarrow{v}^{\circ})^{\bullet} \underrightarrow{\omega}^{ba\bullet} \times \underrightarrow{v} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{\bullet} \\
&= \underrightarrow{v}^{\circ\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{\circ}(\underrightarrow{\omega}^{ba\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{\omega}^{ba}) \times \underrightarrow{v} + \underrightarrow{\omega}^{ba} \times (\underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}) \\
&= \underrightarrow{v}^{\circ\circ} + 2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba\circ} \times \underrightarrow{v} + \underrightarrow{\omega}^{ba} \times (\underrightarrow{\omega}^{ba} \times \underrightarrow{v})
\end{aligned}$$

  - Notice that $\underrightarrow{\omega}^{ba\bullet} = \underrightarrow{\omega}^{ba\circ}$
  - If we interpret $\underrightarrow{v}$ as a position vector, then $2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{\circ}$ would be the Coriolis acceleration, $\underrightarrow{\omega}^{ba} \times (\underrightarrow{\omega}^{ba} \times \underrightarrow{r})$ would be the centripetal acceleration and $\underrightarrow{\omega}^{ba\circ} \times \underrightarrow{r}$ would be the tangential acceleration (or Euler acceleration)
- In coordinate form, the velocity transport equation is $\dot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\dot{\boldsymbol{v}}_b + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{v}_b)$
- For acceleration this is $\ddot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\ddot{\boldsymbol{v}}_b + 2\boldsymbol{\omega}_b^{ba\times} \dot{\boldsymbol{v}}_b + \dot{\boldsymbol{\omega}}_b^{ba\times} \boldsymbol{v} + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{\omega}_b^{ba\times} \boldsymbol{v}_b)$
- How do rotation matrices change with rotating reference frames?
  - $\underrightarrow{\boldsymbol{\mathcal{F}}}_b^{T\bullet} = \underrightarrow{\boldsymbol{\mathcal{F}}}_a^T \dot{\boldsymbol{C}}_{ab}$ since $\underrightarrow{\boldsymbol{\mathcal{F}}}_a^{T\bullet} = \boldsymbol{0}$
  - $\boldsymbol{\omega}_b^{ba\times} = \underrightarrow{\boldsymbol{\mathcal{F}}}_b \cdot \underrightarrow{\boldsymbol{\mathcal{F}}}_b^{T\bullet} = \boldsymbol{C}_{ba} \underrightarrow{\boldsymbol{\mathcal{F}}}_a \cdot \underrightarrow{\boldsymbol{\mathcal{F}}}_a^T \dot{\boldsymbol{C}}_{ab} = \boldsymbol{C}_{ba} \dot{\boldsymbol{C}}_{ab}$
  - If we transpose both sides we get $-\boldsymbol{\omega}_b^{ba\times} = \dot{\boldsymbol{C}}_{ba} \boldsymbol{C}_{ab}$
  - Multiplying by $\boldsymbol{C}_{ba}$ and rearranging, we get $\dot{\boldsymbol{C}}_{ba} + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{C}_{ba} = \boldsymbol{0}$
    * We have found a differential equation for the rotation matrix that describes its evolution
    * This is known as *Poisson's kinematical equation*
- Consider now 3 reference frames $\underrightarrow{\boldsymbol{\mathcal{F}}}_a, \underrightarrow{\boldsymbol{\mathcal{F}}}_b, \underrightarrow{\boldsymbol{\mathcal{F}}}_c$; what is the relationship among $\underrightarrow{\omega}^{ba}, \underrightarrow{\omega}^{cb}$ and $\underrightarrow{\omega}^{ca}$?
  - $\dot{\boldsymbol{C}}_{ca} = \dot{\boldsymbol{C}}_{cb} \boldsymbol{C}_{ba} + \boldsymbol{C}_{cb} \dot{\boldsymbol{C}}_{ba}$
  - $\boldsymbol{\omega}_c^{ca\times} = -\dot{\boldsymbol{C}}_{ca} \boldsymbol{C}_{ac}$

$$\begin{aligned}
&= -\dot{\boldsymbol{C}}_{cb} \boldsymbol{C}_{ba} \boldsymbol{C}_{ab} \boldsymbol{C}_{bc} - \boldsymbol{C}_{cb} \dot{\boldsymbol{C}}_{ba} \boldsymbol{C}_{ab} \boldsymbol{C}_{bc} \\
&= -\dot{\boldsymbol{C}}_{cb} \boldsymbol{C}_{bc} - \boldsymbol{C}_{cb} \dot{\boldsymbol{C}}_{ba} \boldsymbol{C}_{ab} \boldsymbol{C}_{bc} \\
&= \boldsymbol{\omega}_c^{cb\times} + \boldsymbol{C}_{cb} \boldsymbol{\omega}_b^{ba\times} \boldsymbol{C}_{bc} \\
&= \boldsymbol{\omega}_c^{cb\times} + (\boldsymbol{C}_{cb} \boldsymbol{\omega}_b^{ba})^{\times} \\
&= (\boldsymbol{\omega}_c^{cb} + \boldsymbol{C}_{cb} \boldsymbol{\omega}_b^{ba})^{\times}
\end{aligned}$$

  - Therefore $\boldsymbol{\omega}_c^{ca} = \boldsymbol{\omega}_c^{cb} + \boldsymbol{C}_{cb} \boldsymbol{\omega}_b^{ba}$
  - If we multiply both sides by $\underrightarrow{\boldsymbol{\mathcal{F}}}_c^T$, we see that $\underrightarrow{\omega}^{ca} = \underrightarrow{\omega}^{cb} + \underrightarrow{\omega}^{ba}$

Important

While angular velocities can be added directly as $\underrightarrow{\omega}^{ca} = \underrightarrow{\omega}^{cb} + \underrightarrow{\omega}^{ba}$, angular accelerations cannot!

Summary

The transport equations relate velocity and acceleration as measured in one frame to how they're measured in another frame:

- $\underrightarrow{v}^{\bullet} = \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}$
- $\dot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\dot{\boldsymbol{v}}_b + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{v}_b)$
- $\underrightarrow{v}^{\bullet\bullet} = \underrightarrow{v}^{\circ\circ} + 2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba\circ} \times \underrightarrow{v} + \underrightarrow{\omega}^{ba} \times (\underrightarrow{\omega}^{ba} \times \underrightarrow{v})$
- $\ddot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\ddot{\boldsymbol{v}}_b + 2\boldsymbol{\omega}_b^{ba\times} \dot{\boldsymbol{v}}_b + \dot{\boldsymbol{\omega}}_b^{ba\times} \boldsymbol{v} + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{\omega}_b^{ba\times} \boldsymbol{v}_b)$

Rotation Representations Revisited

- Given an axis-angle representation $\boldsymbol{a}, \phi$, the Euler parameters are $\eta = \cos\frac{1}{2}\phi$ and $\boldsymbol{\varepsilon} = \boldsymbol{a}\sin\frac{1}{2}\phi$
  - η and $\boldsymbol{\varepsilon}$ are not independent, because we stipulate that $\eta^2 + \boldsymbol{\varepsilon}^T\varepsilon = 1$
  - Euler parameters don't have a singularity
  - These are also known as quaternions
- Consider a 1-2-3 set of Euler angles, so $\boldsymbol{C} = \boldsymbol{C}_3\boldsymbol{C}_2\boldsymbol{C}_1$
  - $\boldsymbol{\omega}^\times = -\dot{\boldsymbol{C}}\boldsymbol{C}^T$ (Note here $\boldsymbol{\omega} = \boldsymbol{\omega}_b^{ba}$)
  - $\dot{\boldsymbol{C}} = \boldsymbol{C}_3\boldsymbol{C}_2\dot{\boldsymbol{C}}_1 + \boldsymbol{C}_3\dot{\boldsymbol{C}}_2\boldsymbol{C}_1 + \dot{\boldsymbol{C}}_3\boldsymbol{C}_2\boldsymbol{C}_1$
  - $$\begin{aligned}\boldsymbol{\omega}^\times &= -\boldsymbol{C}_3\boldsymbol{C}_2\dot{\boldsymbol{C}}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \boldsymbol{C}_3\dot{\boldsymbol{C}}_2\boldsymbol{C}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \dot{\boldsymbol{C}}_3\boldsymbol{C}_2\boldsymbol{C}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T \\ &= -\boldsymbol{C}_3\boldsymbol{C}_2\dot{\boldsymbol{C}}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \boldsymbol{C}_3\dot{\boldsymbol{C}}_2\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \dot{\boldsymbol{C}}_3\boldsymbol{C}_3^T \\ &= \boldsymbol{C}_3\boldsymbol{C}_2\mathbf{1}_1^\times\dot{\theta}_1\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \boldsymbol{C}_3\mathbf{1}_2^\times\dot{\theta}_2\boldsymbol{C}_3^T - \mathbf{1}_3^\times\dot{\theta}_3 \\ &= (\boldsymbol{C}_3\boldsymbol{C}_2\mathbf{1}_1\dot{\theta}_1)^\times + (\boldsymbol{C}_3\mathbf{1}_2\dot{\theta}_2)^\times + \mathbf{1}_3^\times\dot{\theta}_3\end{aligned}$$
    - Note that $\dot{\boldsymbol{C}}_1\boldsymbol{C}_1^T$ is the angular velocity of a rotation about the first axis, so it is equal to $\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}^\times = \mathbf{1}_1^\times\dot{\theta}_1$
  - This means $\boldsymbol{\omega} = \boldsymbol{C}_3\boldsymbol{C}_2\mathbf{1}_1\dot{\theta}_1 + \boldsymbol{C}_3\mathbf{1}_2\dot{\theta}_2 + \mathbf{1}_3\dot{\theta}_3 = \begin{bmatrix}\boldsymbol{C}_3\boldsymbol{C}_2\mathbf{1}_1 & \boldsymbol{C}_3\mathbf{1}_2 & \mathbf{1}_3\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\\\dot{\theta}_3\end{bmatrix} = \boldsymbol{S}(\theta_2, \theta_3)\dot{\boldsymbol{\theta}}$
    - This is the kinematic equation for a 1-2-3 set of Euler angles (page 89 lists the $\boldsymbol{S}$ matrices for other combinations)
  - At a singularity, $\boldsymbol{S}$ becomes singular (hence the name singularity) – given $\boldsymbol{\omega}$, at a singularity, we cannot find $\dot{\boldsymbol{\theta}}$
- Given any rotation representation, we can write a kinematic equation for it