Lecture 4, Sep 19, 2023

Solving the Sundial

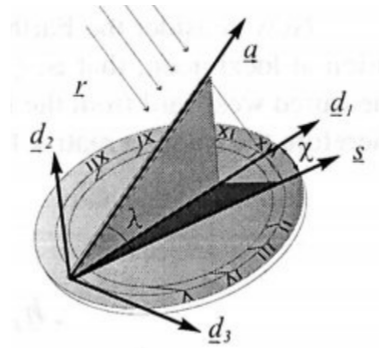

Figure 1: A sundial.

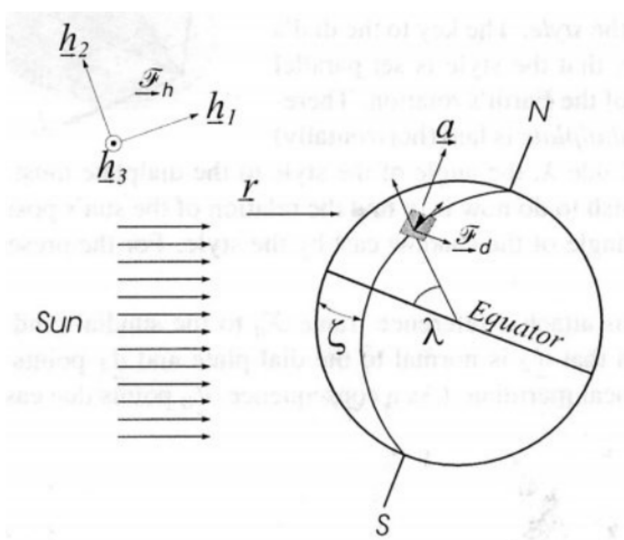

Figure 2: The geometry of the sundial.

- A sundial's upper edge, the *style*, is made parallel to the Earth's axis of rotation, so the sun casts a shadow on the plane of the sundial; how should we draw the markings on the sundial to indicate time?
- We will establish a coordinate system with $\underrightarrow{d}_1, \underrightarrow{d}_3$ being in the plane of the sundial
  - The shadow cast by the style is $\underrightarrow{s} = s_1\underrightarrow{d}_1 + s_2\underrightarrow{d}_3$
  - The angle the shadow makes with $\underrightarrow{d}_1$ is χ, so $\tan\chi = \dfrac{s_3}{s_1}$
  - The shadow $\underrightarrow{s}$, style $\underrightarrow{a}$, and the sun's rays $\underrightarrow{r}$ are in the same plane, so we have $\underrightarrow{s} = c\underrightarrow{a} + \underrightarrow{r}$ (since we don't care about the magnitude, we only need 1 coefficient)
  - Therefore $s_1\underrightarrow{d}_1 + s_3\underrightarrow{d}_3 = c\underrightarrow{a} + \underrightarrow{r} = \underrightarrow{\mathcal{F}}_d^T \begin{bmatrix} s_1 \\ 0 \\ s_3 \end{bmatrix} = s_1\mathbf{1}_1 + s_3\mathbf{1}_3$
- We define another frame, the heliocentric frame $\underrightarrow{\mathcal{F}}_h$, which is aligned with $\underrightarrow{\mathcal{F}}_d$ at noon (when the rotation angle of the earth, $\zeta = 0$)
  - Consider the coordinate system formed by $\underrightarrow{a}, \underrightarrow{h}_3 \times \underrightarrow{a}$ and $\underrightarrow{h}_3$
  - The angle α between the sun's rays $\underrightarrow{r}$ and the style/Earth axis $\underrightarrow{a}$ is seasonally dependent
    - We can get break down $\underrightarrow{r}$ into $\underrightarrow{a}$ and a perpendicular component: $\underrightarrow{r} = \cos\alpha\underrightarrow{a} - \sin\alpha\underrightarrow{h}_3 \times \underrightarrow{a}$

    * $\underrightarrow{a} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \boldsymbol{a} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}$ where λ is the latitude
  - $\underrightarrow{r} = \underrightarrow{\boldsymbol{\mathcal{F}}}_h^T \boldsymbol{r}_h = \underrightarrow{\boldsymbol{\mathcal{F}}}_h^T \left(\cos\alpha \boldsymbol{a} - \sin\alpha \boldsymbol{1}_3^\times \boldsymbol{a}\right)$
- When the Earth rotates, the frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_d$ rotates about the axis $\underrightarrow{a}$ by ζ, so $\boldsymbol{C}_{dh} = \cos\zeta \boldsymbol{1} + (1 - \cos\zeta)\boldsymbol{a}\boldsymbol{a}^T - \sin\zeta \boldsymbol{a}^\times$
- Now we have $\underrightarrow{a}$ and $\underrightarrow{r}$ in frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_d$, we can use the equation before: $c\underrightarrow{a} + \underrightarrow{r} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \begin{bmatrix} s_1 \\ 0 \\ s_3 \end{bmatrix}$
  - This gives us 3 equations, when solved we get $\tan\chi = \dfrac{s_3}{s_1} = \sin\lambda \tan\zeta$
- In general, we want to express everything in the same frame to solve a problem; some vectors are more easily expressed in certain frames than others

Rotation Representations

- We can represent any rotation with a rotation matrix, but it is overspecified since there are 9 components
- Using Euler's theorem, we can completely specify a rotation by the axis-angle pair $(\boldsymbol{a}, \phi)$, which has 3 components only (since $\boldsymbol{a}$ is normalized)
- We can also perform a series of 3 principal axis rotations; these are known as the Euler angles
  - Any sequence of 3 axes works, as long as you don't have a sequence of 2 consecutive identical rotations
  - e.g. $\boldsymbol{C} = \boldsymbol{C}_3(\theta_3)\boldsymbol{C}_1(\theta_2)\boldsymbol{C}_3(\theta_1)$ is a 3-1-3 set of rotations
    * The 3-1-3 set is the one that Euler used
  - In total there are $3 \times 2 \times 2 = 12$ different sets of rotations
  - Given $\boldsymbol{C}$, we can't always find a unique sequence of Euler angles that make up the rotation; this is referred to as a *singularity*
    * e.g. with the 3-1-3 set, if $\theta_2 = 0$ then we won't be able to distinguish θ_3 from θ_1
    * This is why in aerospace typically 1-2-3 or 3-2-1 is used; however any sequence has a singularity, it's just for some sequences the singularity occurs further from the reference point

Kinematics

- Kinematics is the geometry of motion, with no regard for the laws of nature
- Rates of change like $\underrightarrow{v}$ depend on the frame of reference that they are viewed from; when we take a derivative we must note the reference frame

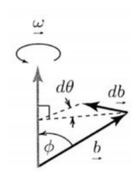

Figure 3: Vector of a fixed length in rotation.

- Consider a vector $\underrightarrow{b}$ of fixed length rotating about a fixed axis $\underrightarrow{a}$ with some rate $\dot{\theta}$
  - Since $\underrightarrow{b}$ is constant length, $\mathrm{d}\underrightarrow{b}$ is normal to $\underrightarrow{b}$ and $\underrightarrow{a}$, so $\mathrm{d}\underrightarrow{b} \propto \underrightarrow{a} \times \underrightarrow{b}$
  - Suppose that in time $\mathrm{d}t$, $\underrightarrow{b}$ rotated $\mathrm{d}\theta$; then $\|\mathrm{d}\underrightarrow{b}\| = \|\underrightarrow{b}\| \sin\phi \mathrm{d}\theta$
  - If we let $\underrightarrow{\omega} = \underrightarrow{a}\dot{\theta}$, then $\mathrm{d}\theta = \|\underrightarrow{\omega}\|\,\mathrm{d}t$ and so $\|\mathrm{d}\underrightarrow{b}\| = \|\underrightarrow{\omega} \times \underrightarrow{b}\|\,\mathrm{d}t$
  - Therefore we have $\mathrm{d}\underrightarrow{b} = \underrightarrow{\omega} \times \underrightarrow{b}\,\mathrm{d}t$ and so $\left.\dfrac{\mathrm{d}\underrightarrow{b}}{\mathrm{d}t}\right|_{\underrightarrow{\mathcal{F}}_a} = \underrightarrow{\omega} \times \underrightarrow{b}$
- We will use the notation that $\underrightarrow{b}_1^{\bullet} = \left.\dfrac{\mathrm{d}\underrightarrow{b}_1}{\mathrm{d}t}\right|_{\underrightarrow{\mathcal{F}}_a}$ and $\underrightarrow{b}_1^{\circ} = \left.\dfrac{\mathrm{d}\underrightarrow{b}_1}{\mathrm{d}t}\right|_{\underrightarrow{\mathcal{F}}_b}$
- Since the axes of reference frames have constant length, we can use this for all 3 axis vectors

- We denote $\underset{\rightarrow}{\mathcal{F}}_b^{T^\bullet} = \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\mathcal{F}}_b^T = \begin{bmatrix} \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{b}_3 \end{bmatrix}$
    - The $\underset{\rightarrow}{\omega}^{ba}$ is the angular velocity of $\underset{\rightarrow}{\mathcal{F}}_b$ with respect to $\underset{\rightarrow}{\mathcal{F}}_a$
- $\underset{\rightarrow}{\mathcal{F}}_b^{T^\bullet} = \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\mathcal{F}}_b^T = \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{\omega}^{ba^\times} \implies \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T^\bullet} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{\omega}^{ba^\times} \implies \boldsymbol{\omega}^{ba^\times} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T^\bullet}$

- $\boldsymbol{\omega}^{ba^\times} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T^\bullet}$ becomes our definition for angular velocity of frame b with respect to a in general