Lecture 23, Dec 5, 2023

Exam Review

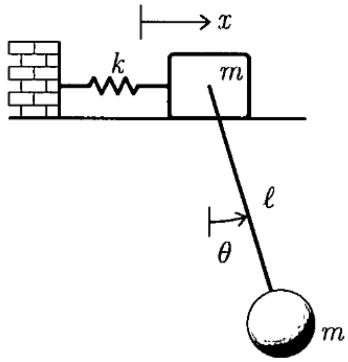

Figure 1: Example problem 1.

- Consider a system with a mass m on a horizontal frictionless plane, connected to inertial space by a spring of stiffness $k = \frac{mg}{l}$, with x measured from the equilibrium position; a pendulum of length l and mass m is connected to the mass, with θ measured from vertical
  a. Derive the potential energy for the system
     - $V = \frac{1}{2}kx^2 - mgl\cos\theta$
     - Approximate to second order: $V = \frac{1}{2}kx^2 - mgl\left(1 - \frac{1}{2}\theta^2\right) = \frac{1}{2}kx^2 - mgl + \frac{1}{2}mgl\theta^2$
     - We want this in the form of $\frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}$, ignoring constant terms
     - Let us define $\boldsymbol{q} = \begin{bmatrix} x \\ l\theta \end{bmatrix}$ so that the units are consistent
     - $\boldsymbol{K} = \begin{bmatrix} k & 0 \\ 0 & \frac{mg}{l} \end{bmatrix} = k\mathbf{1}$ (note no 1/2 in the matrix, since the factor is outside)
  b. Derive the kinetic energy for the system
     - The velocity of the bob is the vector sum of the block's velocity and the bob's velocity relative to the block
     - By the cosine law: $v^2 = \dot{x}^2 + l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos(\pi - \theta) = \dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta$
     - $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mv^2 = \frac{1}{2}m\left(2\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right)$
     - We want this in the form of $\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$
     - To second order: $T = \frac{1}{2}m(2\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta})$
     - $\boldsymbol{M} = m\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (note no l terms in the matrix since these are in $\boldsymbol{q}$ itself)
  c. What are the linearized equations of motion?
     - $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \mathbf{0}$
     - Notice that $\boldsymbol{K}$ is symmetric and positive definite, so all modes are stable (purely imaginary eigenvalues)
  d. Determine the frequencies of vibration
     - This requires us to solve for the eigenvalues
     - $\det(\lambda^2\boldsymbol{M} + \boldsymbol{K}) = 0$
     - Since we know λ are purely imaginary, let $\lambda^2 = -\omega^2$
     - $\det(-\omega^2\boldsymbol{M} + \boldsymbol{K}) = \det\left(\begin{bmatrix} k - 2m\omega^2 & -\omega^2 m \\ -\omega^2 m & k - m\omega^2 \end{bmatrix}\right) = 0$

- Let $\mu^2 = \frac{\omega^2 m}{k}$, then $\det\left(\begin{bmatrix} -2\mu^2 + 1 & -\mu^2 \\ -\mu^2 & -\mu^2 + 1 \end{bmatrix}\right) = 0$ (after dividing through by k)
- $\mu^4 - 2\mu^2 + 1 = 0 \implies \mu^2 = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$
- $\omega^2 = \frac{k}{m}\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right) \implies \omega_1 = \sqrt{\frac{k}{m}\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)}, \omega_2 = \sqrt{\frac{k}{m}\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)}$

e. Determine and sketch the mode shapes of vibration
   - This requires us to solve for the eigenvectors
   - Plug in ω^2 to $(-\omega_\alpha^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{q}_\alpha = \boldsymbol{0}$
   - $\boldsymbol{q}_1 \propto \begin{bmatrix} -1 + \sqrt{5} \\ 3 - \sqrt{5} \end{bmatrix}, \boldsymbol{q}_2 \propto \begin{bmatrix} -1 - \sqrt{5} \\ 3 + \sqrt{5} \end{bmatrix}$
   - The first node has both the block and pendulum on the same side, while the second node has the block and pendulum on different sides in opposing motion

- In general it usually helps to make the coordinates dimensionally consistent, so that the mass and stiffness matrices are dimensionally consistent, which usually makes the math easier

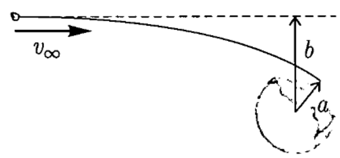

Figure 2: Example problem 2.

- Consider a meteor approaching from infinity with $v_\infty = \sqrt{\frac{\mu}{a}}$, where a is the radius of the Earth and μ is the reduced mass of the meteor and Earth; let the perpendicular distance from the centre of the Earth to the asymptotes of the hyperbolic orbit be b; what would b be if the meteor were to just skim the surface of the Earth?
  - We know this orbit will be hyperbolic, since if it were parabolic, we'd have $v_\infty = 0$
  - The specific energy is $e = \frac{1}{2}v^2 - \frac{\mu}{r}$
  - The specific angular momentum at infinity is $h = bv_\infty$ (since b is the moment arm, and v_∞ is the velocity)
  - When skimming the Earth, we have $h = av_p$, but due to conservation of angular momentum we have $av_p = bv_\infty$ so $b = \frac{av_p}{v_\infty}$
  - To get v_p we use energy conservation: at infinity $e = \frac{1}{2}v_\infty^2$ (since $r \to \infty$); when skimming the Earth, $e = \frac{1}{2}v_p^2 - \frac{\mu}{a}$
  - Therefore $v_p^2 = 2\left(\frac{1}{2}v_\infty^2 + \frac{\mu}{a}\right) = \frac{3\mu}{a} \implies v_p = \sqrt{3}v_\infty$
  - Therefore $b = \frac{a\sqrt{3}v_\infty}{v_\infty} = \sqrt{3}a$

- Consider a uniform hoop of mass m and radius a rolling without slipping on an incline of angle γ; the distance travelled by the hoop is x and its rotation angle is ϕ

  a. What is the constraint in Pffafian form?
     - $\mathrm{d}x - a\,\mathrm{d}\phi = 0$

  b. Derive the equations of motion using Lagrange multipliers and solve for the translational acceleration down the incline
     - $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}m(\dot{x}^2 + a^2\dot{\phi}^2)$
     - For the hoop, $I = ma^2$ since all the mass is concentrated at a radius of a
     - $V = mgh = -mgx\sin\gamma$

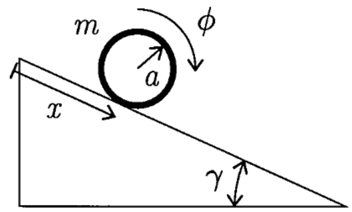

Figure 3: Example problem 3.

- No virtual work since the constraint forces do no work
- In the form $\Xi_1\,\mathrm{d}x + \Xi_2\,\mathrm{d}\phi = 0$ we have $\Xi_1 = 1, \Xi_2 = -a$
- $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}, \dfrac{\partial L}{\partial x} = mg\sin\gamma, \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) = ma^2\ddot{\phi}, \dfrac{\partial L}{\partial \phi} = 0$
- Recall: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{q}_k}\right) - \dfrac{\partial L}{\partial q_k} = \sum_j \lambda_j \Xi_{jk}$ so the equations of motion are:
  * $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = \lambda\Xi_1 \implies m\ddot{x} - mg\sin\gamma = \lambda$
  * $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = \lambda\Xi_2 \implies ma^2\ddot{\phi} = -a\lambda$
  * $\Xi_1\,\mathrm{d}x + \Xi_2\,\mathrm{d}\phi = 0 \implies \dot{x} - a\dot{\phi} = 0$ which we can integrate
- Solving gives $\ddot{x} = \dfrac{1}{2}g\sin\gamma$