Lecture 22, Nov 30, 2023

Modal Analysis

- Consider first the unforced equation of motion, $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}$
- Substituting in the test solution $\boldsymbol{q}_0 e^{\lambda t}$ gives us $(\lambda^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{q}_0 = \boldsymbol{0}$
  - Clearly, this is satisfied for the trivial solution $\boldsymbol{q}_0 = \boldsymbol{0}$, but we want some non-quiescent solution
  - For $\boldsymbol{q}_0$ to be nonzero, we need $\det(\lambda^2 \boldsymbol{M} + \boldsymbol{K}) = 0$; this resembles an eigenproblem where instead of identity we have $\boldsymbol{M}$
  - There will be multiple such λ and $\boldsymbol{q}_0$
- λ_α^2 are then the eigenvalues and $\boldsymbol{q}_\alpha$ the eigenvectors; $\det(\lambda^2\boldsymbol{M} + \boldsymbol{K}) = 0$ is the characteristic equation or eigenequation
  - In general the eigenequation gives us an n-th order polynomial in λ^2
  - Consider multiplying both sides by the Hermitian of $\boldsymbol{q}_\alpha$: $\lambda_\alpha^2 \boldsymbol{q}_\alpha^H \boldsymbol{M}\boldsymbol{q}_\alpha + \boldsymbol{q}_\alpha^H \boldsymbol{K}\boldsymbol{q}_\alpha = 0$
    * $\boldsymbol{M}$ is real and symmetric, so $\boldsymbol{q}_\alpha^H \boldsymbol{M}\boldsymbol{q}_\alpha$ is real; furthermore its positive-definiteness means this is also always greater than 0
    * $\boldsymbol{K}$ is real and symmetric, so $\boldsymbol{q}_\alpha^H \boldsymbol{K}\boldsymbol{q}_\alpha$ is also real
    * $\lambda_\alpha^2 = -\dfrac{\boldsymbol{q}_\alpha^H \boldsymbol{K}\boldsymbol{q}_\alpha}{\boldsymbol{q}_\alpha^H \boldsymbol{M}\boldsymbol{q}_\alpha}$ so indeed λ is real
    * Further, $\boldsymbol{K} > 0 \implies \lambda_\alpha^2 < 0$, or $\lambda_\alpha = \pm j\omega_\alpha$
    * By extension, the $\boldsymbol{q}_\alpha$ are real
- Do $\boldsymbol{q}_\alpha$ form a basis?
  - Consider $\lambda_\alpha^2 \boldsymbol{q}_\beta^T \boldsymbol{M}\boldsymbol{q}_\alpha + \boldsymbol{q}_\beta^T \boldsymbol{K}\boldsymbol{q}_\alpha = 0$ and $\lambda_\beta^2 \boldsymbol{q}_\alpha^T \boldsymbol{M}\boldsymbol{q}_\beta + \boldsymbol{q}_\alpha^T \boldsymbol{K}\boldsymbol{q}_\beta = 0$
  - Subtracting the two equations gives $(\lambda_\alpha^2 - \lambda_\beta^2)\boldsymbol{q}_\alpha^T \boldsymbol{M}\boldsymbol{q}_\beta = 0$ (note we can do this since the terms are scalars)
  - Then $\boldsymbol{q}_\alpha^T \boldsymbol{M}\boldsymbol{q}_\beta = \begin{cases} > 0 & \lambda_\alpha^2 = \lambda_\beta^2 \\ 0 & \lambda_\alpha^2 \neq \lambda_\beta^2 \end{cases}$
  - WLOG normalize the $\boldsymbol{q}$ vectors with respect to $\boldsymbol{M}$, then $\boldsymbol{q}_\alpha^T \boldsymbol{M}\boldsymbol{q}_\beta = \delta_{\alpha\beta} \implies \boldsymbol{Q}^T\boldsymbol{M}\boldsymbol{Q} = \boldsymbol{1}$ where $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{q}_1 & \cdots & \boldsymbol{q}_m\end{bmatrix}$
    * Note we might get repeated eigenvalues, but we can always diagonalize due to the symmetry of $\boldsymbol{M}$
  - Plugging back into the first equation, $\lambda_\alpha^2 \delta_{\alpha\beta} + \boldsymbol{q}_\alpha^T \boldsymbol{K}\boldsymbol{q}_\beta^T \implies \boldsymbol{q}_\alpha^T \boldsymbol{K}\boldsymbol{q}_\beta^T = \begin{cases} -\lambda_\alpha^2 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$
  - So we can also write $\boldsymbol{Q}^T\boldsymbol{K}\boldsymbol{Q} = -\Lambda^2$
- Let $\boldsymbol{q}(t) = \displaystyle\sum_{\beta=1}^{n} \boldsymbol{q}_\beta \eta_\beta(t)$
  - $$\begin{aligned} &\sum_{\beta=1}^{n} \boldsymbol{M}\boldsymbol{q}_\beta \ddot{\eta}_\beta + \sum_{\beta=1}^{n} \boldsymbol{K}\boldsymbol{q}_\beta \eta_\beta = \boldsymbol{f} \\ \implies &\sum_{\beta=1}^{n} \boldsymbol{q}_\alpha^T \boldsymbol{M}\boldsymbol{q}_\beta \ddot{\eta}_\beta + \sum_{\beta=1}^{n} \boldsymbol{q}_\alpha^T \boldsymbol{K}\boldsymbol{q}_\beta \eta_\beta = \boldsymbol{q}_\alpha^T \boldsymbol{f} \\ \implies &\sum_{\beta=1}^{n} \delta_{\alpha\beta} \ddot{\eta}_\beta + \sum_{\beta=1}^{n} -\lambda_\alpha^2 \eta_\beta = \boldsymbol{q}_\alpha^T \boldsymbol{f} \\ \implies &\ddot{\eta}_\alpha - \lambda_\alpha^2 \eta_\alpha = f_\alpha \end{aligned}$$
  - We have uncoupled the system of differential equations
  - $\boldsymbol{q}_\beta$ are the *mode shapes*, and $\eta_\beta(t)$ are the *modal coordinates*; $\boldsymbol{q}_\alpha \eta_\alpha$ is a *mode* of vibration
  - Note that if $\boldsymbol{K}$ were positive definite, we would get all negative λ_α^2, giving oscillatory motion; then $\omega_\alpha^2 = -\lambda_\alpha^2$ are the *natural frequencies* of vibration

Double Pendulum Revisited

- Recall: $\boldsymbol{M} = ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \boldsymbol{K} = mgl \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
- Clearly $\boldsymbol{K}$ is positive definite here, so let's write λ^2 as $-\omega^2$
- We want to solve $\det(-\omega_\alpha^2 \boldsymbol{M} + \boldsymbol{K}) = 0$
- Let $\mu_\alpha^2 = -\omega_\alpha^2 \frac{l}{g}$, so that $\det \begin{bmatrix} -2\mu_\alpha^2 + 2 & -\mu_\alpha^2 \\ -\mu_\alpha^2 & -\mu_\alpha^2 + 1 \end{bmatrix} = 0$
- Expanded: $\mu^3 - 4\mu^2 + 2 = 0 \implies \mu^2 = 2 \pm \sqrt{2}$
- Therefore the modal frequencies are $\omega_1 = \sqrt{(2-\sqrt{2})\frac{g}{l}}, \omega_2 = \sqrt{(2+\sqrt{2})\frac{g}{l}}$
- Solve the eigenequation to get $\frac{\theta_{1,2}}{\theta_{1,1}} = \sqrt{2}$ and $\frac{\theta_{2,2}}{\theta_{2,1}} = -\sqrt{2}$
- For each of the modes, at any time, the ratio of the coordinates remains the same
- Notice that in the second mode, we have a *node* – a point that does not move
  - In general, for an n degree of freedom system, we will have n modes; mode n will have $n-1$ nodes

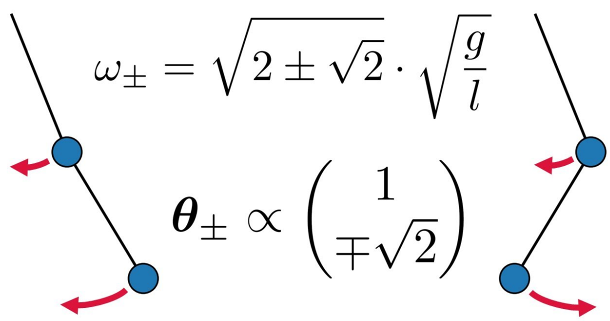

Figure 1: Vibrational modes of the double pendulum.