Lecture 21, Nov 28, 2023

Vibrations: Equation of Motion

- Consider a system of N rigid bodies and particles, described by a set of n generalized coordinates q_k
- Consider a potential V, then the forces are given by $f = -\dfrac{\partial V}{\partial q_k}$, which are zero at equilibrium
- WLOG choose the equilibria to be when $q_k = 0$, then we can expand the potential about the equilibrium:
  - $V(\boldsymbol{q}) = V_0 + \sum_k \dfrac{\partial V}{\partial q_k} q_k + \dfrac{1}{2} \sum_{k,j} \dfrac{\partial^2 V}{\partial q_j \partial q_k} q_j q_k$
  - We can take $V_0 = 0$ since in general the reference potential level does not matter; at an equilibrium we also have $\dfrac{\partial V}{\partial q_k} = 0$
  - Therefore $V = \dfrac{1}{2} \sum_{k,j} \dfrac{\partial^2 V}{\partial q_j \partial q_k} q_j q_k$
- We may express $V = \dfrac{1}{2} \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q}$ for small disturbances
  - $\boldsymbol{K}$ is a matrix of second partials, known as the *stiffness matrix*
  - Due to symmetry of second partials, $\boldsymbol{K}$ is symmetric (but note it is not necessarily definite)
- For kinetic energy, $T = \dfrac{1}{2} \sum_{i=1}^{N} \left(m_i \boldsymbol{v}_i^T \boldsymbol{v}_i + \boldsymbol{\omega}_i^T \boldsymbol{I}_i \boldsymbol{\omega}_i \right)$
  - $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i(q_1, \dots, q_k) = \sum_{k=1}^{n} \dfrac{\partial \boldsymbol{r}_i}{\partial q_k} \dot{q}_k = \sum_{k=1}^{n} \boldsymbol{a}_{ik} \dot{q}_k$
  - $\boldsymbol{\omega}_i^\times = -\dot{\boldsymbol{C}}_i \boldsymbol{C}_i^T = \sum_k -\dfrac{\partial \boldsymbol{C}_i}{\partial q_k} \boldsymbol{C}_i^T \dot{q}_k = \sum_k \boldsymbol{b}_{ik}^\times \dot{q}_k \implies \boldsymbol{\omega}_i = \sum_k \boldsymbol{b}_{ik} \dot{q}_k$
  - We will assume that both have no dependence on $\boldsymbol{q}$
  - Therefore $T = \dfrac{1}{2} \sum_{j,k} \left[\sum_i m_i \boldsymbol{a}_{ij}^T \boldsymbol{a}_{ik} \dot{q}_j \dot{q}_k + \sum_i \boldsymbol{b}_{ij}^T \boldsymbol{I}_i \boldsymbol{b}_{ik} \dot{q}_j \dot{q}_k \right]$
- Let $M_{jk} = \sum_i m_i \boldsymbol{a}_{ij}^T \boldsymbol{a}_{ij} + \sum_i \boldsymbol{b}_{ij}^T \boldsymbol{I} \boldsymbol{b}_{ij}$, then $T = \dfrac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k = \dfrac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}}$
  - $\boldsymbol{M}$ is symmetric and positive definite, because for any nonzero $\dot{q}$, we expect some kind of positive kinetic energy
- The non-conservative forces are $\widehat{\delta W}_\triangle = \sum_k f_k \delta q_k = \delta \boldsymbol{q}_k^T \boldsymbol{f}$
- Using Hamilton's principle, we seek to find $\delta \int_{t_1}^{t_2} L \, \mathrm{d}t - \int_{t_1}^{t_2} \widehat{\delta W}_\triangle \, \mathrm{d}t = 0$
  - $$\begin{aligned} \delta \int_{t_1}^{t_2} L \, \mathrm{d}t - \int_{t_1}^{t_2} \widehat{\delta W}_\triangle \, \mathrm{d}t &= \int_{t_1}^{t_2} \left[\delta \left(\frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} - \frac{1}{2} \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q} \right) + \widehat{\delta W}_\triangle \right] \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\left(\delta \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} - \delta \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q} \right) + \delta \boldsymbol{q}^T \boldsymbol{f} \right] \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left(-\delta \boldsymbol{q}^T \boldsymbol{M} \ddot{\boldsymbol{q}} - \delta \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q} + \delta \boldsymbol{q}^T \boldsymbol{f} \right) \mathrm{d}t \\ &= \int_{t_1}^{t_2} \delta \boldsymbol{q}^T \left(-\boldsymbol{M} \ddot{\boldsymbol{q}} - \boldsymbol{K} \boldsymbol{q} + \boldsymbol{f} \right) \mathrm{d}t \end{aligned}$$
  - Note we used integration by parts and eliminated the boundary term as in the derivation for the Euler-Lagrange equation
  - Setting this to zero, we get that the term inside the brackets must be zero
- Therefore the equation of motion is $\boldsymbol{M} \ddot{\boldsymbol{q}} + \boldsymbol{K} \boldsymbol{q} = \boldsymbol{f}(t)$
  - Notice the similarity to the 1 dimensional spring-mass system $m\ddot{x} + kx = f$
  - If we had linear damping, we could add a $\boldsymbol{D} \dot{q}$ term, where $\boldsymbol{D}$ is symmetric and positive semi-definite

  - We could also add a $\boldsymbol{G}\dot{\boldsymbol{q}}$ term, where $\boldsymbol{G}$ is a skew-symmetric matrix representing gyric effects
  - Finally we can add a $\boldsymbol{H}\boldsymbol{q}$ term where $\boldsymbol{H}$ is a skew-symmetric matrix representing circulatory effects (follower forces, e.g. lift and drag)
  - This is the general form for a linear system
- Note that to obtain the linear system, we need to find the kinetic and potential energies to second order

Example: Double Pendulum

- Consider a double pendulum with masses $m_1 = m_2 = m$, angles θ_1, θ_2 from vertical, and link lengths $l_1 = l_2 = l$
- $T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$
  - $v_1 = l_1\dot{\theta}_1$
  - For v_2, we need to add the velocities of the first mass and the second mass relative to the first mass, which are in general not in the same direction
  - The relative speed is $v_2' = l_2\dot{\theta}_2$, which forms a triangle with v_1
  - Using the cosine law: $v_2^2 = v_1^2 + (v_2')^2 - 2v_1v_2'\cos(\pi - (\theta_2 - \theta_1)) = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$
  - $T = \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right)$
    * We can expand $\cos(\theta_2 - \theta_1) = 1 - \frac{1}{2}(\theta_2 - \theta_1)^2$ to second order
    * However since we already have a $\dot{\theta}_1\dot{\theta}_2$ multiplying this, it will be 4th order, which we can ignore
  - Therefore $T = \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right)$
- This gives $T = \frac{1}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{M}\dot{\boldsymbol{\theta}}$ where $\boldsymbol{M} = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix} = ml^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
- $V = mgl(1 - \cos\theta_1) + mgl(1 - \cos\theta_2 + 1 - \cos\theta_1)$
  - Expanding this to second order, we get $\frac{1}{2}mgl(2\theta_1^2 + \theta_2^2)$
  - Therefore $V = \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{K}\boldsymbol{\theta}$ where $\boldsymbol{K} = mgl\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
- The equation of motion is therefore $\boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{K}\boldsymbol{\theta} = \boldsymbol{0}$