Lecture 20, Nov 23, 2023

Analysis of a Spinning Top

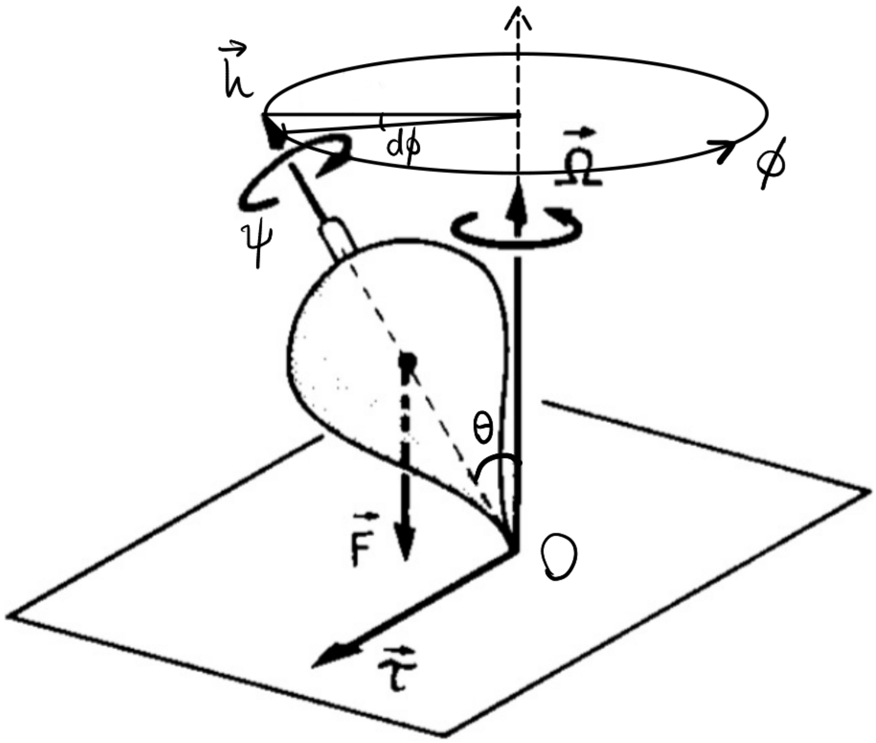

Figure 1: Precession of a spinning top.

- Consider a general spinning top spinning about its axis of symmetry, with the contact point fixed; what is the rate of the precession of $\underset{\rightarrow}{h}$ about the vertical axis?
- In general we have 2 contributors to $\underset{\rightarrow}{h}$: the spin of the top itself and the wobbling; we will assume that the top is spinning fast enough that almost all $\underset{\rightarrow}{h}$ lies in the spinning of the top itself
- We also assume the angular velocity has constant magnitude
- We want $\dfrac{\mathrm{d}\phi}{\mathrm{d}t}$, where $\mathrm{d}\phi$ is a small change to the angle of the projection of $\underset{\rightarrow}{h}$ onto the horizontal plane
  - $\mathrm{d}\phi = \dfrac{\mathrm{d}h}{h\sin\theta}$ where θ is the angle made with the vertical axis (note θ is called the *nutation angle*)
- $h = J_a\nu$ where ν is the spin rate and J_a is the axial moment of inertia
- The change in $\underset{\rightarrow}{h}$ is due to the only externally applied force – gravity
  - Gravity acts at the centre of mass, which is a distance r from O
  - Therefore it exerts a torque $\underset{\rightarrow}{\tau} = m\underset{\rightarrow}{g}r\sin\theta = \dot{\underset{\rightarrow}{h}}$
  - The magnitude is then $\tau = mgr\sin\theta = \dfrac{\mathrm{d}h}{\mathrm{d}t}$
- Substitute relevant quantities: $\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = \dfrac{\frac{\mathrm{d}h}{\mathrm{d}t}}{J_a\nu\sin\theta} = \dfrac{mgr\sin\theta}{J_a\nu\sin\theta} = \dfrac{mgr}{J_a\nu}$
  - This is the *precession rate*
- To analyze the full motion, we will use the Lagrangian formulation; consider a 3-1-3 set of Euler angles ϕ, θ, ψ where the 3-axis is the vertical axis
  - Recall: $\boldsymbol{\omega} = \boldsymbol{S}(\theta, \psi)\begin{bmatrix}\dot\phi \\ \dot\theta \\ \dot\psi\end{bmatrix} = \begin{bmatrix}\dot\phi\sin\theta\sin\phi + \dot\theta\cos\psi \\ \dot\theta\sin\theta\cos\psi - \theta\sin\psi \\ \dot\phi\cos\theta + \dot\psi\end{bmatrix}$

- The kinetic energy is $T = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{J}\boldsymbol{\omega}$, where $\boldsymbol{J} = \begin{bmatrix} J_t & & \\ & J_t & \\ & & J_a \end{bmatrix}$ (assuming symmetry)
  - * We will call the transverse moments of inertial J_t and the axial one J_a
- Expanding this out: $T = \frac{1}{2}J_t(\dot{\theta}^2 + \phi^2 \sin^2\theta) + \frac{1}{2}J_a(\dot{\phi}\cos\theta + \dot{\psi})^2$
- The potential energy is taken at the centre of mass and so $V = mgr\cos\theta$

- Notice L is not dependent on ϕ and ψ, so $\frac{\partial L}{\partial \dot{\phi}}$ and $\frac{\partial L}{\partial \dot{\psi}}$ are constant (these are the angular momenta about the two axes)
  - Note these are called *cyclic* or *ignorable* coordinates
  - $\frac{\partial L}{\partial \dot{\phi}} = J_t\dot{\phi}\sin^2\theta + J_a\cos\theta(\dot{\phi}\cos\theta + \dot{\psi}) = J_t\omega_\phi$
  - $\frac{\partial L}{\partial \dot{\psi}} = J_a(\dot{\phi}\cos\theta + \dot{\psi}) = J_a\nu$
  - With the assumption that $\dot{\phi} \ll \dot{\psi}$ we can write the first equation as $J_t\dot{\phi}\sin^2\theta + J_a\nu\cos\theta = J_t\omega_\phi$
- Finally the equation in θ: $J_t\ddot{\theta} + (J_a - J_t)\phi^2\sin\theta\cos\theta + J_a\dot{\phi}\dot{\psi}\sin\theta - mgr\sin\theta = 0$
  - We see that even with no non-conservative forces, the nutation angle still changes
  - The faster that the top is spinning, the less the nutation changes, which is why normally it seems almost constant to us

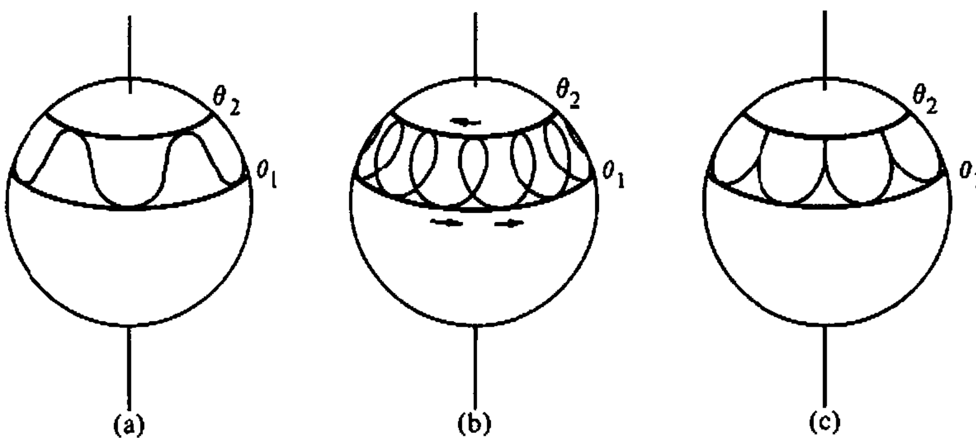

Figure 2: Types of precession of the axis of the top.