Lecture 2, Sep 12, 2023

Frames of Reference

- In this course, we restrict our analysis to dextral (right-handed) orthonormal frames of reference/bases
- Any vector $\underset{\rightarrow}{v}$ can be expressed in any basis: $\underset{\rightarrow}{v} = v_1\underset{\rightarrow}{a}_1 + v_2\underset{\rightarrow}{a}_2 + v_3\underset{\rightarrow}{a}_3$
  - v_1, v_2, v_3 are the coordinates of $\underset{\rightarrow}{v}$ in the basis $\underset{\rightarrow}{a}$
- As an alternative notation, consider $\underset{\rightarrow}{v} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v} = v_1\underset{\rightarrow}{a}_1 + v_2\underset{\rightarrow}{a}_2 + v_3\underset{\rightarrow}{a}_3$, where $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} = \begin{bmatrix} \underset{\rightarrow}{a}_1 \\ \underset{\rightarrow}{a}_2 \\ \underset{\rightarrow}{a}_3 \end{bmatrix}$
  - $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}$ is a matrix of vectors, so we call it a *vectrix*
  - We can also write this as $\underset{\rightarrow}{v} = \boldsymbol{v}^T\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}$ which gives the same result
- We can now define various operations using vectrix notation:
  - $\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} = (\boldsymbol{u}^T\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}) \cdot (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v}) = \boldsymbol{u}^T(\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T)\boldsymbol{v} = \boldsymbol{u}^T\mathbf{1}\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v}$
    * Note that $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T = \begin{bmatrix} \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{a}_1 & \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{a}_2 & \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{a}_3 \\ \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{a}_1 & \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{a}_2 & \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{a}_3 \\ \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{a}_1 & \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{a}_2 & \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{a}_3 \end{bmatrix} = \mathbf{1}$ because the basis in $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}$ is orthonormal
    * This definition is consistent with our usual definition of an inner product
  - $\underset{\rightarrow}{u} \times \underset{\rightarrow}{v} = (\boldsymbol{u}^T\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}) \times (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v}) = \boldsymbol{u}^T(\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \times \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T)\boldsymbol{v} = (u_2v_3 - u_3v_2)\underset{\rightarrow}{a}_1 + (u_3v_1 - u_1v_3)\underset{\rightarrow}{a}_2 + (u_1v_2 - u_2v_1)\underset{\rightarrow}{a}_3$
    * Note that $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \times \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T = \begin{bmatrix} \underset{\rightarrow}{a}_1 \times \underset{\rightarrow}{a}_1 & \underset{\rightarrow}{a}_1 \times \underset{\rightarrow}{a}_2 & \underset{\rightarrow}{a}_1 \times \underset{\rightarrow}{a}_3 \\ \underset{\rightarrow}{a}_2 \times \underset{\rightarrow}{a}_1 & \underset{\rightarrow}{a}_2 \times \underset{\rightarrow}{a}_2 & \underset{\rightarrow}{a}_2 \times \underset{\rightarrow}{a}_3 \\ \underset{\rightarrow}{a}_3 \times \underset{\rightarrow}{a}_1 & \underset{\rightarrow}{a}_3 \times \underset{\rightarrow}{a}_2 & \underset{\rightarrow}{a}_3 \times \underset{\rightarrow}{a}_3 \end{bmatrix} = \begin{bmatrix} 0 & \underset{\rightarrow}{a}_3 & -\underset{\rightarrow}{a}_2 \\ -\underset{\rightarrow}{a}_3 & 0 & \underset{\rightarrow}{a}_1 \\ \underset{\rightarrow}{a}_2 & -\underset{\rightarrow}{a}_1 & 0 \end{bmatrix}$ where we have assumed right-handedness
    * Also note that $\underset{\rightarrow}{u} \times \underset{\rightarrow}{v} = -\underset{\rightarrow}{v} \times \underset{\rightarrow}{u} \iff \boldsymbol{u}^\times\boldsymbol{v} = -\boldsymbol{v}^\times\boldsymbol{u}$, i.e. the cross product is anti-commutative
    * This definition is also consistent with our usual definition for cross product
    * Alternatively $\underset{\rightarrow}{u} \times \underset{\rightarrow}{v} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{u}^\times\boldsymbol{v}$ where $\boldsymbol{u}^\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$
      - $\boldsymbol{u}^\times$ is also known as the skew-symmetric form of $\boldsymbol{u}$ since $(\boldsymbol{u}^\times)^T = -\boldsymbol{u}^\times$
      - Also note that as expected $\det(\boldsymbol{u}^\times) = 0$ (in fact the determinant of any odd-dimensioned skew-symmetric matrix is zero)
  - $\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} \times \underset{\rightarrow}{w} = \boldsymbol{u}^T\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \cdot (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v}^\times\boldsymbol{w}) = \boldsymbol{u}^T\boldsymbol{v}^\times\boldsymbol{w}$
    * This is called the triple product and represents the volume of a parallelepiped formed by the 3 vectors
  - $\underset{\rightarrow}{u} \times (\underset{\rightarrow}{v} \times \underset{\rightarrow}{w}) = \boldsymbol{u}^T\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \times (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v}^\times\boldsymbol{w}) = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{u}^\times\boldsymbol{v}^\times\boldsymbol{w}$
  - $(\underset{\rightarrow}{u} \times \underset{\rightarrow}{v}) \times \underset{\rightarrow}{w} = (\boldsymbol{u}^\times\boldsymbol{v})^T\underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \times \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T\boldsymbol{w} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T(\boldsymbol{u}^\times\boldsymbol{v})^\times\boldsymbol{w}$
    * Notice that this is in general not the same as the result above, so the cross product is not associative
- Consider two frames a and b, then $\underset{\rightarrow}{v} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T\boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T\boldsymbol{v}_b$; how do we relate $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$?
  - $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T\boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T\boldsymbol{v}_b \implies \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T\boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T\boldsymbol{v}_b \implies \boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T\boldsymbol{v}_b = \boldsymbol{C}_{ab}\boldsymbol{v}_b$
  - $\boldsymbol{C}_{ab}$ is our transformation matrix from b to a
  - Expanded: $\boldsymbol{C}_{ab} = \begin{bmatrix} \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{b}_3 \\ \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{b}_3 \\ \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{b}_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta_{11}) & \cos(\theta_{12}) & \cos(\theta_{13}) \\ \cos(\theta_{21}) & \cos(\theta_{22}) & \cos(\theta_{23}) \\ \cos(\theta_{31}) & \cos(\theta_{32}) & \cos(\theta_{33}) \end{bmatrix}$
  - Note some properties of $\boldsymbol{C}_{ab}$:
    * $\boldsymbol{C}_{ab}\boldsymbol{C}_{ba} = \boldsymbol{C}_{aa} = \mathbf{1}$, so $\boldsymbol{C}_{ab} = \boldsymbol{C}_{ba}^{-1}$
    * $\boldsymbol{C}_{ab} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T = (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T)^T = \boldsymbol{C}_{ba}$
    * Therefore $\boldsymbol{C}_{ab}^{-1} = \boldsymbol{C}_{ab}^T$
    * Since $\boldsymbol{C}^T\boldsymbol{C} = \mathbf{1}$ for all rotation matrices, $\det \boldsymbol{C}^2 = 1$ so $\det \boldsymbol{C} = 1$
      - Note the determinant of $\boldsymbol{C}$ can be negative when going from a left-handed to a right-handed frame and vice versa but we will not consider these in this course