Lecture 19, Nov 21, 2023

Spin Stability of Rigid Bodies

- Consider a system with no external torque, spinning at a constant nominal rate $\boldsymbol{\omega} = \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix}$; under what conditions is this system stable?
- Consider a small perturbation, such that $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega}(t)$ where $\Delta\boldsymbol{\omega}(t) = \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \end{bmatrix}$
- Plugging into Euler's equations: $\begin{cases} I_1\Delta\dot{\omega}_1 - (I_2 - I_3)(\nu + \Delta\omega_1)\Delta\omega_3 = 0 \\ I_2\Delta\dot{\omega}_2 - (I_3 - I_1)\Delta\omega_3\Delta\omega_1 = 0 \\ I_3\Delta\dot{\omega}_3 - (I_1 - I_2)(\nu + \Delta\omega_2) = 0 \end{cases}$
  - After linearizing: $\begin{cases} I_1\Delta\dot{\omega}_1 - (I_2 - I_3)\nu\Delta\omega_3 = 0 \\ I_2\Delta\dot{\omega}_2 = 0 \\ I_3\Delta\dot{\omega}_3 - (I_1 - I_2)\nu\Delta\omega_1 = 0 \end{cases}$
  - $\Delta\omega_2$ is then constant, so it is always stable
  - Taking the derivative of the first equation and substituting in $\Delta\dot{\omega}_3$, we can get a differential equation for $\Delta\omega_1$
  - $\Delta\ddot{\omega}_1 + \dfrac{(I_2 - I_3)(I_2 - I_1)}{I_1 I_3}\nu^2\Delta\omega_1 = 0 \implies \Delta\ddot{\omega}_1 + \beta^2\Delta\omega_1 = 0$
    - Note we could do the same to the second equation and we would get something in the exact same form, with the same β^2
  - This is now an oscillator, so for stability we need $\beta^2 > 0$; $\beta^2 \leq 0$ makes it unstable
- For $\beta^2 > 0$ we need $I_2 - I_3$ and $I_2 - I_1$ to have the same sign, for a rotation about the 2 axis to be stable
  - This requires either $I_2 > I_1, I_3$ or $I_2 < I_1, I_3$ – it has to be the major (largest inertia) or minor (smallest inertia) axis, but not the intermediate axis
- From the equation of motion: $\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times\boldsymbol{I}\boldsymbol{\omega} = 0 \implies \boldsymbol{\omega}^T\boldsymbol{I}\dot{\boldsymbol{\omega}} = 0 \implies \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega}\right) = 0$
  - Note the $\boldsymbol{\omega}^T\boldsymbol{\omega}^\times$ cancels
  - Integrating this, we get $\dfrac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega} = T$ is a constant – this is the rotational kinetic energy
  - In the principal axis frame, expanding this out we get $\dfrac{\omega_1^2}{\frac{2T}{I_1}} + \dfrac{\omega_2^2}{\frac{2T}{I_2}} + \dfrac{\omega_3^2}{\frac{2T}{I_3}} = 0$
  - Geometrically, means that $\boldsymbol{\omega}$ must lie on the surface of an ellipsoid – the energy ellipsoid
- Multiplying instead by $\boldsymbol{\omega}^T\boldsymbol{I}$, we have $\boldsymbol{\omega}^T\boldsymbol{I}^2\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega}^\times\boldsymbol{I}\boldsymbol{\omega} = 0 \implies \boldsymbol{\omega}^T\boldsymbol{I}^2\dot{\boldsymbol{\omega}} = 0$
  - Note $\boldsymbol{z}^T\boldsymbol{\omega}^\times\boldsymbol{z} = 0$ for any skew-symmetric $\boldsymbol{\omega}^\times$ (since it is a scalar, and if you transpose it you get its negative)
  - Doing the same and integrating gives us $\boldsymbol{\omega}^T\boldsymbol{I}^2\boldsymbol{\omega} = h^2$, another constant – this is the square of the angular momentum
  - $\dfrac{\omega_1^2}{\frac{h^2}{I_1^2}} + \dfrac{\omega_2^2}{\frac{h^2}{I_2^2}} + \dfrac{\omega_3^2}{\frac{h^2}{I_3^2}} = 0$
  - Geometrically this gives us yet another ellipsoid for $\boldsymbol{\omega}$ – the momentum ellipsoid
- Since $\boldsymbol{\omega}$ has to be on both ellipsoids, it must stay on their intersection – these intersection lines are called *polhodes*
  - Due to the squaring of I, the momentum ellipsoid will usually appear more stretched out than the energy ellipsoid
  - Specifying the energy and angular momentum initial conditions choose a pair of polhodes; a third initial condition is needed to solve for the angular momentum as a function of time
  - For a pure spin about the minor axis, the momentum ellipsoid is entirely contained within the energy ellipsoid, so the only intersections are the top and bottom points

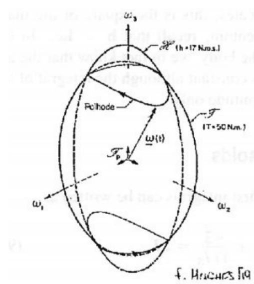

Figure 1: Intersection of the energy and momentum ellipsoids.

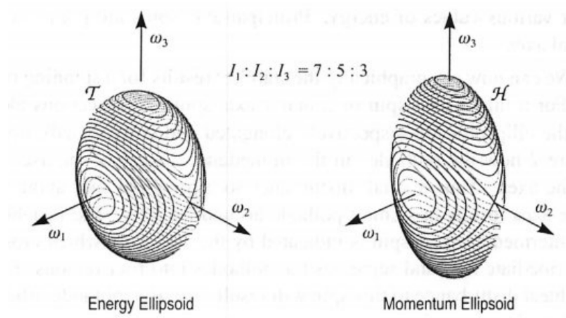

Figure 2: The energy and momentum ellipsoids with polhodes.

    * With a small perturbation, the momentum ellipsoid increases slightly in size, so we get a polhode near the pole, which is a small circle; since the angular momentum stays within the circle, this means we are stable
  - For a pure spin about the major axis, the energy ellipsoid is entirely contained within the momentum ellipsoid
- Notice $T = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{I}\boldsymbol{\omega}$ so $\boldsymbol{\nabla}_\omega T = \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{h}$
  - Therefore $\boldsymbol{h}$ is always normal to the energy ellipsoid, and moreover it is fixed in inertial space since there are no external torques
  - We can interpret this as the energy ellipsoid "rolling" on the *invariable plane*
    * This is because $T = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{h}$ is constant, so the projection of $\underset{\rightarrow}{\omega}$ onto $\underset{\rightarrow}{h}$ is constant – the tip of $\underset{\rightarrow}{\omega}$ must lie on a plane normal to $\underset{\rightarrow}{h}$
  - As we roll the ellipsoid, $\boldsymbol{\omega}$ remains on both the surface of the energy ellipsoid and the invariable plane
  - This is known as *Poinsot's Geometric Interpretation*
  - The curve traced out on the invariable plane is known as the *herpolhode*
  - Since the energy ellipsoid exists in the principal axis frame, which is a body-fixed frame, the motion of the energy ellipsoid is the motion of the body
- In real life however, since any body dissipates energy, $\dot{T} < 0$, so a minor axis spin will slowly shift towards a major axis spin; as long as the system can lose energy, minor axis spins are unstable
  - The major axis spin is asymptotically stable since the energy and momentum ellipsoids must intersect, so at this point the energy ellipsoid can't spin more

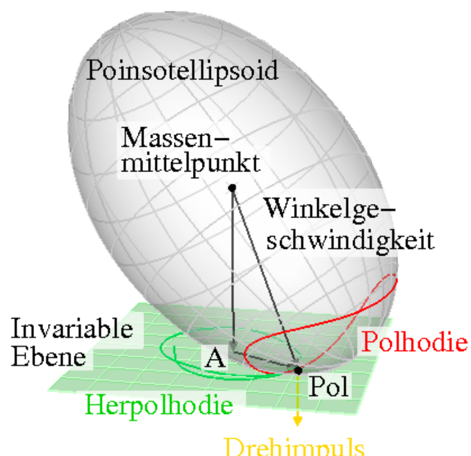

Figure 3: Poinsot's construction, in German for some reason.