Lecture 18, Nov 16, 2023

Dynamics of Rigid Bodies

Momentum of Rigid Bodies

- Consider a rigid body R and some inertial reference point $O_{\mathfrak{I}}$
  - Each differential mass element $\mathrm{d}m$ has momentum $\mathrm{d}\underrightarrow{p} = \underrightarrow{r}^{\bullet}\,\mathrm{d}m$
  - Therefore the overall momentum is $\underrightarrow{p} = \int_R \underrightarrow{r}^{\bullet}\,\mathrm{d}m$
- Consider some reference point O fixed to the body, and let $\underrightarrow{\rho}$ be the position of a mass element relative to O
  - $\underrightarrow{r}^{\bullet} = \underrightarrow{v}_O + \underrightarrow{\rho}^{\bullet} = \underrightarrow{v}_O + \underrightarrow{\rho}^{\circ} + \underrightarrow{\omega} \times \underrightarrow{\rho}$
  - But $\underrightarrow{\rho}$ is fixed as seen in a body-relative frame, so $\underrightarrow{\rho}^{\circ} = 0$ (unless the body is deformable)
  - $\underrightarrow{p} = \int_R (\underrightarrow{v}_O + \underrightarrow{\omega} \times \underrightarrow{\rho})\,\mathrm{d}m = \int_R \underrightarrow{v}_O\,\mathrm{d}m - \int_R \underrightarrow{\rho} \times \underrightarrow{\omega}\,\mathrm{d}m = m\underrightarrow{v}_O - \left(\int_R \underrightarrow{\rho}\,\mathrm{d}m\right) \times \underrightarrow{\omega}$
- Let $\underrightarrow{c}_O = \int_R \underrightarrow{\rho}\,\mathrm{d}m$ be the *first moment of mass* (aka *first moment of inertia*), then $\underrightarrow{p} = m\underrightarrow{v}_O - \underrightarrow{c}_O \times \underrightarrow{\omega}$
  - Note that $\underrightarrow{c}_O$ has a subscript since it is computed with respect to O
  - Expressed in a body frame $\underrightarrow{\mathcal{F}}_b$, $\boldsymbol{p} = m\boldsymbol{v}_O - \boldsymbol{c}_O^{\times}\boldsymbol{\omega}$
    - Note $\boldsymbol{c}_O$ is a constant in $\underrightarrow{\mathcal{F}}_b$
- For angular momentum, $\mathrm{d}\underrightarrow{h}_O = \underrightarrow{\rho} \times \mathrm{d}\underrightarrow{p} = \underrightarrow{\rho} \times (\underrightarrow{v}_O - \underrightarrow{\rho} \times \underrightarrow{\omega})\,\mathrm{d}m \implies \underrightarrow{h}_O = \underrightarrow{c}_O \times \underrightarrow{v}_O - \int_R \underrightarrow{\rho} \times (\underrightarrow{\rho} \times \underrightarrow{\omega})\,\mathrm{d}m$
  - In $\underrightarrow{\mathcal{F}}_b$,
  $$\begin{aligned} \boldsymbol{h}_O &= \boldsymbol{c}_O^{\times}\boldsymbol{v}_O - \int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\boldsymbol{\omega}\,\mathrm{d}m \\ &= \boldsymbol{c}_O^{\times}\boldsymbol{v}_O + \left(-\int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\,\mathrm{d}m\right)\boldsymbol{\omega} \\ &= \boldsymbol{c}_O^{\times}\boldsymbol{v}_O + \boldsymbol{J}_O\boldsymbol{\omega} \end{aligned}$$
  - $\boldsymbol{J}_O = -\int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\,\mathrm{d}m$ is the *second moment of mass*, or the *inertia matrix*
    - Note $\underrightarrow{J}_O$ is a *second order tensor*; in vector form, $\underrightarrow{h}_O = \underrightarrow{c}_O \times \underrightarrow{v}_O + \underrightarrow{J}_O \cdot \underrightarrow{\omega}$
    - $\boldsymbol{J}_O = \int_R (\rho^2\mathbf{1} - \boldsymbol{\rho}\boldsymbol{\rho}^T)\,\mathrm{d}m$
  - Let $\boldsymbol{s}$ be any vector, then
  $$\begin{aligned} \boldsymbol{s}^T\boldsymbol{J}_O\boldsymbol{s} &= -\boldsymbol{s}^T\left(\int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\,\mathrm{d}m\right)\boldsymbol{s} \\ &= -\int_R \boldsymbol{s}^T\boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\boldsymbol{s}\,\mathrm{d}m \\ &= \int_R (\boldsymbol{\rho}^{\times}\boldsymbol{s})^T(\boldsymbol{\rho}^{\times}\boldsymbol{s})\,\mathrm{d}m \\ &= \int_R \|\boldsymbol{\rho}^{\times}\boldsymbol{s}\|^2\,\mathrm{d}m \end{aligned}$$
    - There will always be some $\underrightarrow{\rho}$ that is not parallel to $\underrightarrow{s}$ for any 3D body, so this integral is always positive for a nonzero $\underrightarrow{s}$
    - Therefore $\boldsymbol{J}_O$ is symmetric positive definite (hence why we include a minus sign in the definition)
- A *second-order tensor* $\underrightarrow{D} = \underrightarrow{a}\underrightarrow{b}$ is defined such that $\underrightarrow{D} \cdot \underrightarrow{v} = (\underrightarrow{a}\underrightarrow{b}) \cdot \underrightarrow{v} = \underrightarrow{a}(\underrightarrow{b} \cdot \underrightarrow{v})$
- In matrix form, $\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{h}_O \end{bmatrix} = \begin{bmatrix} m\mathbf{1} & -\boldsymbol{c}_O^{\times} \\ \boldsymbol{c}_O^{\times} & \boldsymbol{J}_O \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_O \\ \boldsymbol{\omega} \end{bmatrix}$, where $\boldsymbol{M} = \begin{bmatrix} m\mathbf{1} & -\boldsymbol{c}_O^{\times} \\ \boldsymbol{c}_O^{\times} & \boldsymbol{J}_O \end{bmatrix}$ is the *mass matrix*, which is also symmetric positive definite
- If we choose $O = \odot$, then $\int_R \underrightarrow{r}\,\mathrm{d}m = 0$, so $\underrightarrow{c} = 0$
  - $\boldsymbol{p} = m\boldsymbol{v}_O$
  - $\boldsymbol{h}_{\odot} = \boldsymbol{J}_{\odot}\boldsymbol{\omega} = \boldsymbol{I}\boldsymbol{\omega}$, where we denote $\boldsymbol{I}$ as the inertia matrix about the centre of mass
- Consider two inertia matrices $\boldsymbol{J}_A, \boldsymbol{J}_B$ relative to points A, B; for a differential mass element, denote

position relative to A by $\underrightarrow{a}$, position relative to B by $\underrightarrow{b}$ and the relative position between A and B is $\underrightarrow{\rho}^{BA}$

- $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{\rho}^{BA}$ in a common body frame (draw this out)
- $$\begin{aligned}\boldsymbol{J}_A &= -\int_R \boldsymbol{a}^\times \boldsymbol{a}^\times \,\mathrm{d}m \\ &= -\int_R (\boldsymbol{b} + \boldsymbol{\rho}^{BA})^\times (\boldsymbol{b} + \boldsymbol{\rho}^{BA}) \,\mathrm{d}m \\ &= -\int_R \left(\boldsymbol{b}^\times \boldsymbol{b}^\times + \boldsymbol{\rho}^{BA\times} \boldsymbol{b}^\times + \boldsymbol{b}^\times \boldsymbol{\rho}^{BA\times} + \boldsymbol{\rho}^{BA\times} \boldsymbol{\rho}^{BA\times}\right) \mathrm{d}m \\ &= -\int_R \boldsymbol{b}^\times \boldsymbol{b}^\times \,\mathrm{d}m - \boldsymbol{\rho}^{BA\times} \int_R \boldsymbol{b}^\times \,\mathrm{d}m - \int_R \boldsymbol{b}^\times \,\mathrm{d}m \boldsymbol{\rho}^{BA\times} - \int \mathrm{d}m \boldsymbol{\rho}^{BA\times} \boldsymbol{\rho}^{BA\times} \\ &= \boldsymbol{J}_B - \boldsymbol{\rho}^{BA\times} \boldsymbol{c}_B^\times - \boldsymbol{c}_B^\times \boldsymbol{\rho}^{BA\times} - m\boldsymbol{\rho}^{BA\times} \boldsymbol{\rho}^{BA\times}\end{aligned}$$
- This is the *parallel axis theorem* for an inertia matrix

Theorem

Parallel Axis Theorem: Given an inertia matrix $\boldsymbol{J}_B$ around a point B, and relative position $\boldsymbol{\rho}^{BA}$ from A to B, we can find the inertia matrix around A, $\boldsymbol{J}_A$ as:

$$\boldsymbol{J}_A = \boldsymbol{J}_B - \boldsymbol{\rho}^{BA\times} \boldsymbol{c}_B^\times - \boldsymbol{c}_B^\times \boldsymbol{\rho}^{BA\times} - m\boldsymbol{\rho}^{BA\times} \boldsymbol{\rho}^{BA\times}$$

- Consider the same reference point in two frames $\underrightarrow{\mathcal{F}}_a, \underrightarrow{\mathcal{F}}_b$; denote $\boldsymbol{J}_a, \boldsymbol{J}_b$ be the inertia matrix about this point expressed in the two frames
  - $$\begin{aligned}\boldsymbol{J}_a &= -\int_R \boldsymbol{\rho}_a^\times \boldsymbol{\rho}_a^\times \,\mathrm{d}m \\ &= -\int_R (\boldsymbol{C}_{ab}\boldsymbol{\rho}_b)^\times (\boldsymbol{C}_{ab}\boldsymbol{\rho}_b)^\times \,\mathrm{d}m \\ &= -\int_R (\boldsymbol{C}_{ab}\boldsymbol{\rho}_b^\times \boldsymbol{C}_{ba})(\boldsymbol{C}_{ab}\boldsymbol{\rho}_b^\times \boldsymbol{C}_{ba}) \,\mathrm{d}m \\ &= \boldsymbol{C}_{ab}\left(-\int_R \boldsymbol{\rho}_b^\times \boldsymbol{\rho}_b^\times\right)\boldsymbol{C}_{ba} \\ &= \boldsymbol{C}_{ab}\boldsymbol{J}_b\boldsymbol{C}_{ba}\end{aligned}$$
  - This is the *rotational transformation theorem* for an inertia matrix
  - Note for a second-order tensor, $\underrightarrow{J} = \underrightarrow{\mathcal{F}}_a^T \boldsymbol{J}_a \underrightarrow{\mathcal{F}}_a \iff \boldsymbol{J}_a = \underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{J} \cdot \underrightarrow{\mathcal{F}}_a^T$, so this identity follows
    * The result applies for any second-order tensor

Motion of Rigid Bodies

- To get the equations of motion, we can treat a rigid body like a grammar of particles
- For a grammar of particles, $\underrightarrow{p}^\bullet = \underrightarrow{f}$ and $\underrightarrow{h}_O^\bullet + \underrightarrow{v}_O \times \underrightarrow{p} = \underrightarrow{\tau}_O$
  - $\boldsymbol{p} = m\boldsymbol{v}_O - \boldsymbol{c}_O^\times \boldsymbol{\omega}$
  - $\boldsymbol{h}_O = \boldsymbol{c}_O^\times \boldsymbol{v}_O + \boldsymbol{J}_O\boldsymbol{\omega}$
  - But to use these, we have to first convert the derivative $(\cdot)^\bullet$ with respect to inertial frame into a derivative with respect to body frame
    * $\underrightarrow{p}^\circ + \underrightarrow{\omega} \times \underrightarrow{p} = \underrightarrow{f}$
    * $\underrightarrow{h}_O^\circ + \underrightarrow{\omega} \times \underrightarrow{h}_O + \underrightarrow{v}_O \times \underrightarrow{p} = \underrightarrow{\tau}_O$
  - In the body frame:
    * $\dot{\boldsymbol{p}} + \boldsymbol{\omega}^\times \boldsymbol{p} = \boldsymbol{f}$
    * $\dot{\boldsymbol{h}}_O + \boldsymbol{\omega}^\times \boldsymbol{h}_O + \boldsymbol{v}_O^\times \boldsymbol{p} = \boldsymbol{\tau}_O$
- Therefore the equations of motion for a rigid body are given by, in the general case:
  - $m\dot{\boldsymbol{v}}_O - \boldsymbol{c}_O^\times \dot{\boldsymbol{\omega}} + m\boldsymbol{\omega}^\times \boldsymbol{v}_O - \boldsymbol{\omega}^\times \boldsymbol{c}_O^\times \boldsymbol{v}_O = \boldsymbol{f}$
  - $\boldsymbol{c}_O^\times \dot{\boldsymbol{v}}_O + \boldsymbol{J}_O \dot{\boldsymbol{\omega}} - \boldsymbol{c}_O^\times \boldsymbol{\omega}^\times \boldsymbol{v}_O + \boldsymbol{\omega}^\times \boldsymbol{J}_O \boldsymbol{\omega} = \boldsymbol{\tau}_O$

- In matrix form, $\boldsymbol{M}\begin{bmatrix}\dot{\boldsymbol{v}}_O\\ \dot{\boldsymbol{\omega}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\omega}^\times & \boldsymbol{0}\\ \boldsymbol{v}_O^\times & \boldsymbol{\omega}^\times\end{bmatrix}\boldsymbol{M}\begin{bmatrix}\boldsymbol{v}_O\\ \boldsymbol{\omega}\end{bmatrix} = \begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{\tau}_O\end{bmatrix}$
  - $\begin{bmatrix}\boldsymbol{v}_O\\ \boldsymbol{\omega}\end{bmatrix}$ is a generalized velocity and $\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{\tau}_O\end{bmatrix}$ is a generalized force
- If $O = ◕$, we can simplify (where all quantities are relative to the centre of mass):
  - $m\dot{\boldsymbol{v}} + m\boldsymbol{\omega}^\times\boldsymbol{v} = \boldsymbol{f}$
  - $\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times\boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}$
- Note that in the general case, the equations of motion are coupled; but if we use ◕, the rotational equation is uncoupled, making it much easier to solve
- Solving this gives us the angular velocity of the body, but not the orientation; for that we need to use Poisson's equation $\dot{\boldsymbol{C}} + \boldsymbol{\omega}^\times\boldsymbol{C} = \boldsymbol{0}$, or for Euler angles $\boldsymbol{\omega} = \boldsymbol{S}\dot{\boldsymbol{\theta}}$ (or axis-angle/quaternion)

* Kinetic energy:
$$\begin{aligned}T &= \frac{1}{2}\int_R \underset{\rightarrow}{r}^{\cdot} \cdot \underset{\rightarrow}{r}^{\cdot}\,\mathrm{d}m\\ &= \frac{1}{2}\int_R (\underset{\rightarrow}{v}_O - \underset{\rightarrow}{\rho} \times \underset{\rightarrow}{\omega})\cdot(\underset{\rightarrow}{v}_O - \underset{\rightarrow}{\rho} \times \underset{\rightarrow}{\omega})\,\mathrm{d}m\\ &= \frac{1}{2}\int_R (\boldsymbol{v}_O - \boldsymbol{\rho}^\times\boldsymbol{\omega})^T(\boldsymbol{v}_O - \boldsymbol{\rho}^\times\boldsymbol{\omega})\,\mathrm{d}m\\ &= \frac{1}{2}m\boldsymbol{v}_O^T\boldsymbol{v}_O - \boldsymbol{v}_O^T\boldsymbol{c}_O^\times\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{J}_O\boldsymbol{\omega}\end{aligned}$$
  - Notice that this has 2 parts: translational, rotational, and a coupling term, which disappears when we use the centre of mass reference frame
  - In matrix form, $\boldsymbol{T} = \frac{1}{2}\begin{bmatrix}\boldsymbol{v}_O\\ \boldsymbol{\omega}\end{bmatrix}^T\boldsymbol{M}\begin{bmatrix}\boldsymbol{v}_O\\ \boldsymbol{\omega}\end{bmatrix}$
* Note we can expand the inertia matrix as $\boldsymbol{I} = \int_R \begin{bmatrix}y^2+z^2 & -xy & -xz\\ -xy & x^2+z^2 & -yz\\ -xz & -yz & x^2+y^2\end{bmatrix}\mathrm{d}m$
  - In general $\boldsymbol{I}$ will be fully populated; but can we diagonalize it?
  - We know $\boldsymbol{I}$ is symmetric positive definite, so it is diagonalizable and the eigenvector matrix is orthogonal
  - There will always be a $\boldsymbol{E}$ such that $\boldsymbol{E}^{-1}\boldsymbol{I}\boldsymbol{E} = \boldsymbol{\Lambda}$; but since $\boldsymbol{E}^{-1} = \boldsymbol{E}^T$ (and choose $\boldsymbol{E}$ such that $\det\boldsymbol{E} = 1$), it is a rotation matrix
  - Recall that the inertia matrix transforms as $\boldsymbol{C}_{ab}\boldsymbol{I}_b\boldsymbol{C}_{ba} = \boldsymbol{I}_a$, which has the exact same form as the diagonalization we found
  - Therefore we can always find a reference frame such that $\boldsymbol{I}$ is diagonal; this is referred to as the *principal-axis frame*
  - This works even if we don't use ◕ as our reference point
* In the principal axis frame, with ◕ as our reference point, the rotational equation reduces to:
  - $I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1$
  - $I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \tau_2$
  - $I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3$
  - These are known as *Euler's equations*

Summary

For a rigid body, let $\boldsymbol{\rho}$ be the position of a differential mass element relative to O in a body-fixed frame, then:

- The *first moment of mass/inertia* $\boldsymbol{c}_O = \int_R \boldsymbol{\rho} \, \mathrm{d}m$, which is zero if $O =$ ◕
- The *second moment of mass/inertia matrix* $\boldsymbol{J}_O = -\int_R \boldsymbol{\rho}^\times \boldsymbol{\rho}^\times \, \mathrm{d}m$, which is diagonal in the principal axis frame and denoted $\boldsymbol{I}$ if $O =$ ◕

Then the linear and angular momenta are given by

$$\boldsymbol{p} = m\boldsymbol{v}_O - \boldsymbol{c}_O^\times \boldsymbol{\omega}, \quad \boldsymbol{h}_O = \boldsymbol{c}_O^\times \boldsymbol{v}_O + \boldsymbol{J}_O \boldsymbol{\omega}$$

The equations of motion are given by, in general,

$$m\dot{\boldsymbol{v}}_O - \boldsymbol{c}_O^\times \dot{\boldsymbol{\omega}} + m\boldsymbol{\omega}^\times \boldsymbol{v}_O - \boldsymbol{\omega}^\times \boldsymbol{c}_O^\times \boldsymbol{v}_O = \boldsymbol{f}$$
$$\boldsymbol{c}_O^\times \dot{\boldsymbol{v}}_O + \boldsymbol{J}_O \dot{\boldsymbol{\omega}} - \boldsymbol{c}_O^\times \boldsymbol{\omega}^\times \boldsymbol{v}_O + \boldsymbol{\omega}^\times \boldsymbol{J}_O \boldsymbol{\omega} = \boldsymbol{\tau}_O$$

Using $O =$ ◕, this reduces to

$$m\dot{\boldsymbol{v}} + m\boldsymbol{\omega}^\times \boldsymbol{v} = \boldsymbol{f}$$
$$\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}$$

And the kinetic energy is given by, in general

$$T = \frac{1}{2} m \boldsymbol{v}_O^T \boldsymbol{v}_O - \boldsymbol{v}_O^T \boldsymbol{c}_O^\times \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{J}_O \boldsymbol{\omega}$$

where the middle coupling term disappears when using $O =$ ◕.