Lecture 17, Nov 14, 2023

Calculus of Variations – The Brachistochrone Problem

- Consider a ball starting at point $A = (x_1, y_1)$, rolling under the influence of gravity to point $B = (x_2, y_2)$; what is the shape of the curve that minimizes the travel time?
- We want to find $y(x)$ such that $y(x_1) = y_1, y(x_2) = y_2$ and the travel time $T = \int_A^B \mathrm{d}t$ is minimized
  - Consider a differential curve element $\mathrm{d}s$; $\frac{\mathrm{d}s}{\mathrm{d}t} = v$ so $\mathrm{d}t = \frac{\mathrm{d}s}{v}$
  - $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$
  - We can find v by conservation of energy: $\frac{1}{2}mv^2 + mgy = E \implies v = \sqrt{2g(y_0 - y)}$
  - The problem becomes:
    * Minimize $\frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{y_0 - y}}\,\mathrm{d}x$ over $y(x)$
    * Subject to $y(x_1) = y_1, y(x_2) = y_2$
- We can generalize this to minimizing $I = \int_{x_1}^{x_2} F(y, y', x)\,\mathrm{d}x$
  - I is a *functional* – a function that takes a function and gives a number
  - How do we minimize with respect to an entire function?
- Let $y^*(x)$ be a minimum, and let $y(x) = y^*(x) + \epsilon\eta(x)$, where ϵ is small and $\eta(x)$ is any function subject to $\eta(x_1) = \eta(x_2) = 0$ (so our boundary conditions are satisfied)
  - $y' = {y^*}' + \epsilon\eta' \implies F(y, y', x) = F(y^* + \epsilon\eta, {y^*}' + \epsilon\eta', x)$
  - $I = \Phi(\epsilon) = \int_{x_1}^{x_2} F(y^* + \epsilon\eta, {y^*}' + \epsilon\eta', x)\,\mathrm{d}x$ and $\epsilon = 0$ must be a minimum, if y^* is a minimum
  - Therefore the necessary condition is $\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0$
- $\frac{\mathrm{d}F}{\mathrm{d}\epsilon} = \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\epsilon} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}\epsilon}$

  $= \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'$
  - $$\begin{aligned}\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)\mathrm{d}x \\ &= \left[\frac{\partial F}{\partial y'}\eta\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,\mathrm{d}x\end{aligned}$$
    * Note we applied integration by parts and used the boundary conditions $\eta(x_1) = \eta(x_2) = 0$
  - Since this is always equal to 0 and η can be anything, we can conclude that the part inside the brackets must always be zero
    * This is can be proven and is known as the *Fundamental Lemma of Variational Calculus*
- Therefore the optimality condition is $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$
  - Notice that this is identical to Lagrange's equation in a conservative system
  - This is the *Euler-Lagrange Equation*
- An alternative way to derive this is to let $\delta y = y - y^*$ (which behaves like $\epsilon\eta$, then we would have $\delta I = 0$; we refer to this as a *stationary value* for I
  - δ is the *variational operator*; think of it as taking the Taylor expansion of a function
    * $\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$
    * Note derivatives, integrals, and δ commute

  * $\delta\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\delta y)$
  * $\delta \int f \,\mathrm{d}x = \int \delta f \,\mathrm{d}x$
- $\delta I = \int_{x_1}^{x_2} \delta F(y, y', x)\,\mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)\mathrm{d}x = 0$
- We can then apply integration by parts and use $\delta y(x_1) = \delta y(x_2) = 0$, and apply the fundamental lemma as normal

Hamiltonian Mechanics

- *Hamilton's Principle*: the motion of a system, under the influence of conservative forces, from time t_1 to t_2, is given by the stationary value of the functional $I = \int_{t_1}^{t_2} L\,\mathrm{d}t$, where $L = T - V$
  - Under non-conservative forces, we instead have $\delta \int_{t_1}^{t_2} L\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W_{\triangle}\,\mathrm{d}t = 0$, where $\delta W_{\triangle}$ is the virtual work done by non-conservative forces
  - This is the *extended Hamilton's principle*
- Hamilton's principle, like Newton's laws and Lagrange's principle, is an equivalent description of classical mechanics; all 3 apply at each instance in time
  - Unlike the other two however, Hamiltonian mechanics looks at an interval of time, while the other two methods look at an instant in time
- The *Hamiltonian* is defined by $H = \sum_k \dot{q}_k p_k - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is the generalized momentum of each coordinate
  - $\mathrm{d}H = \sum_k \left(p_k\,\mathrm{d}\dot{q}_k + \dot{q}_k\,\mathrm{d}p_k - \frac{\partial L}{\partial q_k}\,\mathrm{d}q_k - \frac{\partial L}{\partial \dot{q}_k}\,\mathrm{d}\dot{q}_k\right) - \frac{\partial L}{\partial t}\,\mathrm{d}t$
  - For a conservative system, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \implies \dot{p}_k = \frac{\partial L}{\partial q_k}$
  - $\mathrm{d}H = \sum_k (\dot{q}_k\,\mathrm{d}p_k - \dot{p}_k\,\mathrm{d}q_k) - \frac{\partial L}{\partial t}\,\mathrm{d}t$
    * We have shown that any differential in H is given by differentials in $\boldsymbol{p}$, $\boldsymbol{q}$ and t
    * Therefore $H = H(\boldsymbol{q}, \boldsymbol{p}, t)$ and has no dependence on $\dot{q}$
  - By the chain rule, $\mathrm{d}H = \sum_k \left(\frac{\partial H}{\partial p_k}\,\mathrm{d}p_k + \frac{\partial H}{\partial q_k}\,\mathrm{d}q_k\right) + \frac{\partial H}{\partial t}\,\mathrm{d}t$
    * Therefore $\dot{q}_k = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k}$
    * These are known as *Hamilton's Canonical Equations*, and can serve as an alternative formulation of mechanics