Lecture 15, Oct 26, 2023

Example: Spherical Pendulum

- Consider a pendulum of mass m, length l at an angle θ from the vertical and ϕ from the x axis; what are the equations of motion
- Our generalized coordinates are $q_1 = \phi, q_2 = \theta$
- To get the kinetic energy we break up the velocity into two components
  - $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((l\dot{\theta})^2 + (l\dot{\phi}\sin\theta)^2\right) = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$
- Using the pivot as zero, the potential energy due to gravity is $V = mgl\cos\theta$
- No $Q_{k,\triangle}$ because there are no non-conservative forces in our problem
- $L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta$
- Compute the derivatives:
  - $\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right)ml^2\ddot{\theta}$
  - $\frac{\partial L}{\partial\theta} = ml^2\dot{\phi}^2\sin\theta\cos\theta + mgl\sin\theta$
  - $\frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi}\sin^2\theta$
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = ml^2\ddot{\phi}\sin^2\theta - 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta$
  - $\frac{\partial L}{\partial\phi} = 0$
- The two equations are:
  - $ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0$
  - $ml^2\ddot{\phi}\sin^2\theta - 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0$
- Note because $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = 0$, we can immediately say $\frac{\partial L}{\partial\dot{\phi}}$ is a constant
  - In this case we refer to ϕ as an *ignorable*, or *cyclical coordinate*
  - Physically, $ml^2\dot{\phi}\sin^2\theta$ is the angular momentum about the vertical axis, which is conserved, hence it is a constant
  - $\frac{\partial L}{\partial\dot{q}_k}$ is a *generalized momentum*

Constraints – Method of Lagrange Multipliers

- Consider a hoop of radius a rolling without slipping at an angle ϕ down a ramp, at a distance x
  - In this case we only have 1 independent coordinate – we can't move x and ϕ independently of each other
  - $\mathrm{d}x = a\mathrm{d}\phi \implies x = a\phi + x_0$
  - Starting from the differential form we were able to integrate this to get an expression relating x and ϕ; this is not always possible
  - If we had an expression involving x, we can simply replace it by an expression of ϕ
- Now consider that hoop rolling on a 2D surface, at a direction of θ with respect to the x axis
  - $\mathrm{d}x = a\mathrm{d}\phi\cos\theta, \mathrm{d}y = a\mathrm{d}\phi\sin\theta$
  - We have 2 independent coordinates since the change in θ and ϕ dictate the change in x and y
  - But we can no longer directly integrate our differential constraints since x and y depend on the entire history of θ and ϕ
  - Even though we only have 2 independent coordinates, we cannot write x and y independently of ϕ and θ, so we can't substitute x and y for ϕ and θ anymore
- In general, integrable constraints can be written as $\varphi(x, \phi) = 0$ and are referred to as *holonomic* constraints; if they can't be integrated, they are *non-holonomic*

- Note that inequality constraints are holonomic
- Holonomic constraints tell you "where" you can go – they limit the space of coordinates to a subspace that we can reach
- Non-holonomic constraints tell you "how" you can go – they limit the possible paths we can take through the coordinate space

- Recall that from $\sum_k \left[Q_k - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k}\right]\delta q_k = 0$, we concluded that the part inside the brackets was zero for all k because the δq_k are independent; but if we have dependent coordinates, we can no longer do this
- Consider a series of (linearly independent) constraints in the form $\sum_k \Xi_{jk}\delta q_k = 0$ for $j = 1, \dots, m$
  - We can use the method of *Lagrange multipliers*
  - Multiply each constraint by λ_j, so $\sum_j \lambda_j \sum_k \Xi_{jk}\delta q_k = 0$
  - $\sum_{k=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_{k,\triangle} - \sum_{j=1}^{m}\lambda_j\Xi_{jk}\right]\delta q_k = 0$
  - Without loss of generality, let $q_k, k = 1, \dots, m$ be dependent on $q_k, k = m+1, \dots, n$, which are independent
  - We can choose λ_j such that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle} + \sum_{j=1}^{m}\lambda_j\Xi_{jk}$ for $k = 1, \dots, m$
    - Now $\sum_{k=m+1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_{k,\triangle} - \sum_{j=1}^{m}\lambda_j\Xi_{jk}\right]\delta q_k = 0$
    - But we said that q_k for $k = m+1, \dots, n$ are independent, so we can apply the same argument as before
- $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle} + \sum_{j=1}^{m}\lambda_j\Xi_{jk}$ applies for all coordinates, regardless of independence
  - Note that we have $n+m$ unknowns (the q_k and λ_j), which are matched by our $n+m$ equations (n Lagrange equations and m constraints)
  - In general the constraints are $\sum_{k=1}^{n}\Xi_{jk}\mathrm{d}q_k + \Xi_{jt}\mathrm{d}t = 0$
    - Dividing by $\mathrm{d}t$, $\sum_{k=1}^{n}\Xi_{jk}\dot{q}_k + \Xi_{jt} = 0$, which are our m constraint equations
    - This is known as the *Pfaffian form* of the constraints

Example: Atwood's Machine

- Consider two masses m_1, m_2 hung over a pulley with mass m_p, which is concentrated at the circumference (so we don't need to worry about moment of inertia)
- The height of the masses are z_1, z_2; the pulley is rotating by an angle θ, and let $z_3 = a\theta$ where a is the radius of the pulley
- We have 1 degree of freedom, but we will use all 3 coordinates and use 2 constraints:
  - $z_1 = a\theta = z_3 = -z_2 \implies \begin{cases}\mathrm{d}z_1 - \mathrm{d}z_3 = 0\\ \mathrm{d}z_2 + \mathrm{d}z_3 = 0\end{cases}$
  - Therefore $\Xi_{11} = 1, \Xi_{12} = 0, \Xi_{13} = -1; \Xi_{21} = 0, \Xi_{22} = 1, \Xi_{23} = 1$
- Kinetic and potential energy:
  - $T = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 + \frac{1}{2}m_p\dot{z}_3^2$
  - $V = m_1 g z_1 + m_2 g z_2$
- Derivatives:

- $\frac{\partial L}{\partial \dot{z}_1} = m_1 \dot{z}_1, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_1}\right) = m_1 \ddot{z}_1, \frac{\partial L}{\partial z_1} = -m_1 g$
  - $\frac{\partial L}{\partial \dot{z}_2} = m_2 \dot{z}_2, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_2}\right) = m_2 \ddot{z}_2, \frac{\partial L}{\partial z_2} = -m_2 g$
  - $\frac{\partial L}{\partial \dot{z}_3} = m_p \dot{z}_3, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_3}\right) = m_p \ddot{z}_3, \frac{\partial L}{\partial z_3} = 0$
  - Assume no non-conservative forces
- Lagrange's equations:
  - $m_1 \ddot{z}_1 + m_1 g = \lambda_1 \Xi_{11} + \lambda_2 \Xi_{21} = \lambda_1$
  - $m_2 \ddot{z}_2 + m_2 g = \lambda_1 \Xi_{12} + \lambda_2 \Xi_{22} = \lambda_2$
  - $m_p \ddot{z}_3 = \lambda_1 \Xi_{13} + \lambda_2 \Xi_{23} = -\lambda_1 + \lambda_2$
- Constraint equations:
  - $\dot{z}_1 - \dot{z}_3 = 0$
  - $\dot{z}_2 + \dot{z}_3 = 0$
- We can solve this to get $\ddot{z}_1 = -\ddot{z}_2 = \ddot{z}_3 = -\frac{m_1 - m_2}{m_p + m_1 + m_2} g$
  - $\lambda_1 = \frac{m_p + 2m_2}{m_p + m_1 + m_2} m_1 g$
  - $\lambda_2 = \frac{m_p + 2m_1}{-m_p + m_1 + m_2} m_2 g$
  - The idea of Atwood's machine is that we can choose the masses to be nearly equal, so we get an effective g that's very small
  - λ_1 and λ_2 turn out to be the tension on the two sides of the string
- In general, the Lagrange multipliers have meaning – they are associated with the constraint forces