Lecture 14, Oct 24, 2023

Virtual Work and D'Alembert's Principle

- Consider a grammar of particles in static equilibrium; we will have that $\underset{\rightarrow}{f_i} = \underset{\rightarrow}{0}$; then for any small displacement $\delta\underset{\rightarrow}{r_i} = 0$, so $\sum_i \underset{\rightarrow}{f_i} \cdot \delta\underset{\rightarrow}{r_i} = 0$
  - We define this quantity as the *virtual work* $\widehat{\delta W}$ (note the ligature since work is not path-independent)
  - We call $\delta\underset{\rightarrow}{r_i}$ a *virtual displacement*
- In general we can write $\underset{\rightarrow}{f_i} = \underset{\rightarrow}{f}_{i,app} + \underset{\rightarrow}{f}_{i,\square}$, where $\underset{\rightarrow}{f}_{i,app}$ is an applied force and $\underset{\rightarrow}{f}_{i,\square}$ is a constraint force
  - Then $\widehat{\delta W} = \sum_i \underset{\rightarrow}{f}_{i,app} \cdot \delta\underset{\rightarrow}{r_i} + \sum_i \underset{\rightarrow}{f}_{i,\square} \cdot \delta\underset{\rightarrow}{r_i} = \sum_i \underset{\rightarrow}{f}_{i,app} \cdot \delta\underset{\rightarrow}{r_i}$ because constraint forces do no work, provided that the $\delta\underset{\rightarrow}{r_i}$ are consistent with the geometry of the system
    - This is the *principle of virtual work* and is an assumption that we make
- What about particles in dynamic equilibrium?
  - $\underset{\rightarrow}{f_i} = m\ddot{\underset{\rightarrow}{r}}$
  - So we just need to consider $\underset{\rightarrow}{f_i} - m_i\ddot{\underset{\rightarrow}{r_i}}$ as the total force – according to d'Alembert's principle
  - $\widehat{\delta W} = \sum_i (\underset{\rightarrow}{f}_{i,app} - m_i\ddot{\underset{\rightarrow}{r_i}}) \cdot \delta\underset{\rightarrow}{r_i} = 0$ by the same reasoning and assumptions above, as long as $\delta\underset{\rightarrow}{r_i}$ is consistent with the system's constraints
    - This is known as *d'Alembert's principle* (again)
  - Note that since we no longer have to consider constraint forces we will drop the subscript

Lagrangian Mechanics

- Consider an independent set of coordinates $q_1, q_2, \dots, q_n$ where n is the number of degrees of freedom
  - The coordinates must be complete, i.e. satisfy $\underset{\rightarrow}{r_i} = \underset{\rightarrow}{r_i}(q_{11}, q_{12}, \dots, q_n, t)$; the position of any particle must be expressible in terms of the generalized coordinates
  - These coordinates are called *generalized coordinates*, because the do not have to be Cartesian; instead they can be displacements or angles etc
  - We aim to obtain equations of motions in these generalized coordinates only
- Any permissible virtual displacement can then be given in terms of these coordinates: $\delta\underset{\rightarrow}{r_i} = \sum_k \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}\delta q_k$

  – in this form it is clear that the virtual displacements are permissible
  - Note that even though $\underset{\rightarrow}{r_i}$ can be dependent on time, the virtual displacements $\delta\underset{\rightarrow}{r_i}$ are "frozen" in time; this is why we don't need to consider $\frac{\partial \underset{\rightarrow}{r_i}}{\partial t}$
- $\sum_i \underset{\rightarrow}{f_i} \cdot \delta\underset{\rightarrow}{r_i} - \sum_i m_i\ddot{\underset{\rightarrow}{r_i}} \cdot \delta\underset{\rightarrow}{r_i} = 0$
  - $\sum \underset{\rightarrow}{f_i} \cdot \delta\underset{\rightarrow}{r_i} = \sum_i \sum_k \underset{\rightarrow}{f_i} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}\delta q_k = \sum_k Q_k \delta q_k$ where $Q_k = \sum_i \underset{\rightarrow}{f_i} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}$
    - Q_k are referred to as the *generalized forces*
  - $\sum_i m_i\ddot{\underset{\rightarrow}{r_i}} \cdot \delta\underset{\rightarrow}{r_i} = \sum_i \sum_j m_i\ddot{\underset{\rightarrow}{r_i}} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}\delta q_k = \sum_i \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\underset{\rightarrow}{r_i}} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}\right) - m_i\dot{\underset{\rightarrow}{r_i}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \underset{\rightarrow}{r_i}}{\partial q_i}\right)\right]\delta q_k$
  - Note $\frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} = \frac{\partial \underset{\rightarrow}{v_i}}{\partial \dot{q}_k}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}\right) = \frac{\partial \underset{\rightarrow}{v_i}}{\partial q_k}$ so the stuff in brackets becomes $\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\underset{\rightarrow}{v_i} \cdot \frac{\partial \underset{\rightarrow}{v_i}}{\partial \dot{q}_i}\right) - m_i\underset{\rightarrow}{v_i} \cdot \frac{\partial \underset{\rightarrow}{v_i}}{\partial q_k} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}m_i v_i^2\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{1}{2}m_i\underset{\rightarrow}{v_i} \cdot \underset{\rightarrow}{v_i}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \dot{q}_k}T_i\right) - \frac{\partial}{\partial q_k}T_i$
  - Together we have $\sum_k Q_k \delta q_k - \sum_k \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_k}\left(\sum_i T_i\right)\right) - \frac{\partial}{\partial q_k}\left(\sum_i T_i\right)\right]\delta q_k = 0$
  - $\sum_k \left[Q_k - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k}\right]\delta q_k = 0$

- Since δq_k are all independent, we can choose each one independently and arbitrarily; therefore we need $Q_k \delta q_k - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k} = 0$ for all k
- $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k} = Q_k$
  - $Q_k = \sum_i \underset{\rightarrow}{f_i} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}$
  - Split the forces into conservative and non-conservative parts $\underset{\rightarrow}{f_i} = -\vec{\nabla} V_i + \underset{\rightarrow}{f_{i,\triangle}}$
  - Then $Q_k = -\sum_i \vec{\nabla} V_i \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} + \sum_i f_{i,\triangle} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} = -\frac{\partial V}{\partial q_k} + Q_{k,\triangle}$
    - $Q_{k,\triangle}$ are the generalized non-conservative forces
  - Plugging this back in we get $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} + Q_{k,\triangle}$
    - Assuming V is a function of position only, $\frac{\partial V}{\partial \dot{q}_k} = 0$, so we have $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_k}(T - V)\right) + \frac{\partial}{\partial q_k}(T - V) = Q_{k,\triangle}$
  - Letting $L = T - V$, we get $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle}$
- $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle}$ is *Lagrange's equation* (aka the Euler-Lagrange equation)
  - $Q_{k,\triangle} = \sum_i \underset{\rightarrow}{f_{i,\triangle}} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}$ is the generalized non-conservative force
  - $L = T - V$ is the *Lagrangian*, the difference between the kinetic and potential energies (summed across all particles)
    - Note that when we take the partials with respect to $\dot{q}_k$ and q_k, we treat these two as independent
  - For a system with n degrees of freedom, there are n Lagrange's equations
  - If we have no non-conservative forces, then the equation equals zero
  - Lagrange's equation is equivalent to Newton's laws – we can replace $\underset{\rightarrow}{f} = m\ddot{\underset{\rightarrow}{r}}$ by this formulation to yield identical results
- Note that since we started with Newton's second law, this can only be applied in an inertial frame
  - There is a way around this by considering the potential to be velocity-dependent
- Example: pendulum with length l, mass m and angle θ
  - $q_1 = \theta$
  - $T = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$
  - $V = -mg\cos\theta$ (gravity, with the pivot as reference height)
    - Note it doesn't matter what we take as the reference here because we always take the partial derivative of the Lagrangian, so constant factors in energy disappear as expected
  - Assume that there are no non-conservative forces
  - $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$
  - $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$
  - Plugging in, we get $ml^2\ddot{\theta} - (-mgl\sin\theta) = 0 \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0$, exactly what we get with Newtonian mechanics