Lecture 13, Oct 19, 2023

Perturbation Theory

- Perturbation theory deals with the changes in a function (e.g. an orbit) resulting from a small change
- Let $x(t) = x_0(t) + \Delta x(t)$, where $x(t)$ is some nominal solution and $\Delta x(t)$ is some small disturbance; our goal is to get the response $f(x)$ in terms of Δx only, since we might not have a closed-form expression for f
- Recall the equations of motion: $\begin{cases} f_1(r, \dot{r}, \ddot{r}, \omega, \dot{\omega}) = \ddot{r} - r\omega^2 = -\dfrac{\mu}{r^2} \\ f_2(r, \dot{r}, \ddot{r}, \omega, \dot{\omega}) = r\dot{\omega} + 2\dot{r}\omega = 0 \end{cases}$
- Take some some nominal/reference solutions $r_0(t), \omega_0(t)$ so that $\begin{cases} r(t) = r_0(t) + \Delta r(t) \\ \omega(t) = \omega_0(t) + \Delta\omega(t) \end{cases}$
- Consider a satellite in orbit; we fire thrusters such that at time $t = 0$, we have an instantaneous increase in velocity Δv tangential to the orbit
  - The resulting orbit will be slightly elliptical
- Plug in reference solution to first equation: $(\ddot{r}_0 + \Delta\ddot{r}) - (r_0 \Delta r)(\omega_0 + \Delta\omega)^2 = -\dfrac{\mu}{(r_0 + \Delta r)^2}$
  - Expand and ignore all terms second order and above
  - $(\ddot{r}_0 + \Delta\ddot{r}) - r_0\omega_0^2 + 2r_0\omega_0\Delta\omega + \omega_0^2\Delta r = -\dfrac{\mu}{r_0^2\left(1 + \frac{\Delta r}{r_0}\right)} = -\dfrac{\mu}{r_0^2}\left(1 - 2\dfrac{\Delta r}{r_0}\right)$
  - $\ddot{r}_0 - r_0\omega_0^2 + \Delta\ddot{r} - 2r_0\omega_0\Delta\omega - \omega_0^2\Delta r = -\dfrac{\mu}{r_0^2} + 2\dfrac{\mu}{r_0^3}\Delta r$
  - Compare this with the equation of motion, we can subtract $\ddot{r}_0 - r_0\omega_0^2 = -\dfrac{\mu}{r_0^2}$ from both sides
  - $\Delta\ddot{r} - 2r_0\omega_0\Delta\omega - 3\omega_0^2\Delta r = 0$
- The second equation gives $r_0\Delta\dot{\omega} + 2\omega_0\Delta\dot{r} = 0$, with initial conditions $\Delta r(0) = 0, \Delta\dot{r}(0) = 0, \Delta\omega(0) = \dfrac{\Delta v}{r_0}$
  - Integrate directly: $r_0\Delta\omega + 2\omega_0\Delta r = C = \Delta v$
  - The first equation becomes: $\Delta\ddot{r} + 4\omega_0^2\Delta r - 3\omega_0^2\Delta r = 2\omega_0\Delta v$
  - $\Delta\ddot{r} + \omega_0^2\Delta r = 2\omega_0\Delta v$
  - Particular solution: $\Delta r(t) = \dfrac{2\Delta v}{\omega_0}$
  - Homogeneous solution: $c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$
  - Solve for initial conditions: $\Delta r(t) = \dfrac{2\Delta v}{\omega_0}(1 - \cos(\omega_0 t))$
- The resulting orbit is slightly elliptical, and at $t = 0$ we're at the point on the ellipse closest to the focus

Gravity Boosting/Braking

- Consider a body with velocity $\underset{\to}{v}_p$; a spacecraft of velocity $\underset{\to}{v}_s^-$ approaches it from far away, and we want to know the velocity of the spacecraft $\underset{\to}{v}_s^+$ after escaping the planet's gravity
- Shift into the planet's reference frame; the velocity of the spacecraft in this frame is $\underset{\to}{u}_s^- = -\underset{\to}{v}_p + \underset{\to}{v}_s^-$
- Since the spacecraft is coming from infinity, the orbital shape is effectively hyperbolic
  - The spacecraft will be entering and leaving with the same magnitude of velocity in the planet's frame, $\|\underset{\to}{u}_s^-\| = \|\underset{\to}{u}_s^+\|$
- If we now shift into the sun's reference frame, the spacecraft leaves with velocity $\underset{\to}{v}_s^+ = \underset{\to}{u}_s^+ + \underset{\to}{v}_p$
  - Depending on the direction that the vectors are arranged, $\underset{\to}{v}_s^+$ can be much faster or slower than $\underset{\to}{v}_s^-$
  - If the spacecraft passes behind the planet, it will speed up; if it passes in front of the planet, it will slow down
- This is called gravity boosting or braking (aka the slingshot effect)

Example Problem

- Given that the orbital shape of a body is $r = a\sqrt{\cos 2\theta}$, show that in order to create this orbital shape, the force satisfies $f \propto \frac{1}{r^7}$
- For a general central force we have $\begin{cases} \ddot{r} - r\dot{\theta}^2 = \frac{f}{m} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$
- Now we can simply substitute in r: $\begin{cases} \dot{r} = -\frac{a\dot{\theta}\sin 2\theta}{\sqrt{\cos 2\theta}} \\ \ddot{r} = -\frac{a\ddot{\theta}\sin 2\theta}{\sqrt{\cos 2\theta}} - \frac{a\dot{\theta}\sin 2\theta}{\cos^{\frac{3}{2}} 2\theta} - 2a\dot{\theta}^2\sqrt{\cos 2\theta} \end{cases}$
  - Using the second equation, $\dot{\theta} = \frac{h}{r^2}$ and $\ddot{\theta} = -\frac{2h}{r^3}\dot{r} = \frac{2\dot{\theta}^2 \sin 2\theta}{\cos 2\theta}$
  - This allows us to express everything in terms of r in the first equation