Lecture 12, Oct 17, 2023

Recovering Kepler's Laws

- Using Newton's law of gravitation, we will attempt to recover Kepler's 3 laws
- Consider a grammar of 2 particles a and b; the particles have positions $\vec{r}_a$ and $\vec{r}_b$ relative to the center of mass; let $\vec{r} = \vec{r}_a - \vec{r}_b$ be the vector connecting the two particles
  - Since we have no external forces acting on the system, $m\vec{r}_{◕}^{\bullet\bullet} = \vec{f}_{ext} = \vec{0}$, so ◕ is not accelerating with respect to inertial frame, i.e. the ◕ frame is an inertial frame
- $\vec{f}_a^b = -\dfrac{Gm_am_b}{r^3}\vec{r}$ but also $\vec{f}_a^b = m_a\vec{r}_a^{\bullet\bullet}, \vec{f}_b^a = m_b\vec{r}_b^{\bullet\bullet} = -\vec{f}_a^b$
  - $\vec{r}^{\bullet\bullet} = \vec{r}_a^{\bullet\bullet} - \vec{r}_b^{\bullet\bullet} = \dfrac{1}{m_a}\vec{f}_a^b - \dfrac{1}{m_b}\vec{f}_b^a$
  - Using this we get $\vec{r}^{\bullet\bullet} = -\dfrac{\mu}{r^3}\vec{r}$ where $\mu = G(m_a + m_b)$
- Therefore the relative motion of the bodies is given by $\vec{r}^{\bullet\bullet} = -\dfrac{\mu}{r^3}\vec{r}$
  - This means instead of the motion of two bodies, we can fix b and consider the relative motion of a, with its mass replaced by the reduced mass $m = \dfrac{m_am_b}{m_a + m_b}$
- Consider the total angular momentum (about the centre of mass) $\vec{h} = m_a\vec{r}_a \times \vec{r}_a^{\bullet} + m_b\vec{r}_b \times \vec{r}_b^{\bullet}$
  - $\vec{h}^{\bullet} = m_a\vec{r}_a \times \vec{r}_a^{\bullet\bullet} + m_b\vec{r}_b \times \vec{r}_b^{\bullet\bullet}$ (product rule terms cancel)
  - $\vec{h}^{\bullet} = \vec{r}_a \times \vec{f}_a^b + \vec{r}_b \times \vec{f}_b^a = \vec{r}_a \times (k\vec{r}) + \vec{r}_b \times (k\vec{r}) = 0$
  - Note this is just conservation of angular momentum
- $\vec{r} \cdot \vec{h} = m_a\vec{r} \cdot (\vec{r}_a \times \vec{r}_a^{\bullet}) + m_b\vec{r} \times (\vec{r}_b \times \vec{r}_b^{\bullet}) = 0$
  - Note that $\vec{r}_a \times \vec{r}_a^{\bullet}$ is normal to $\vec{r}_a$, which is parallel to $\vec{r}$; therefore the dot product with $\vec{r}$ is zero for both terms
  - Therefore $\vec{r}$ is always normal to $\vec{h}$, but $\vec{h}$ is constant, so $\vec{r}$ is always in some fixed plane, so now we can reduce the problem down to 2 dimensions
- Since the motion is 2-dimensional, we will use polar coordinates (r, θ) to express $\vec{r}$
- Let the (noninertial) orbital frame $\vec{\mathcal{F}}_o$ such that $\vec{o}_1$ is in the same direction as the vector connecting the two masses and $\vec{o}_2$ is in the plane of motion of $\vec{r}$
  - $\vec{r} = \vec{\mathcal{F}}_o^T \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}, \vec{\omega} = \vec{\mathcal{F}}_o^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$
  - $\vec{r}^{\bullet\bullet} = \vec{r}^{\circ\circ} + 2\vec{\omega} \times \vec{r}^{\circ} + \vec{\omega}^{\circ} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\dfrac{\mu}{r^3}\vec{r}$
  - We can now expand this out in the orbital frame and obtain the equations of motion
- $\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\dfrac{\mu}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$ are the 2-body orbital equations of motion
  - Note the multiplying the second equation by r and integrating gives $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$, which corresponds to conservation of angular momentum (ignoring mass) ($r\dot{\theta}$ being the tangential velocity, with moment arm r)
  - This can be solved by making the substitution $r = \dfrac{1}{u}$, which we leave as an exercise to the reader
- We can also solve the equation of motion using vectors:
  - Starting with the equation of motion, we can cross both sides with $\vec{h}$
  - $\vec{r}^{\bullet\bullet} \times \vec{h} = -\dfrac{\mu}{r^3}\vec{r} \times \vec{h}$
  - $\dfrac{\mathrm{d}}{\mathrm{d}t}(\vec{r}^{\bullet} \times \vec{h}) = -\dfrac{\mu}{r^3}\vec{r} \times (\vec{r} \times \vec{r}^{\bullet}) = -\dfrac{\mu}{r^3}((\vec{r} \cdot \vec{r}^{\bullet})\vec{r} - (\vec{r} \cdot \vec{r})\vec{r}^{\bullet}) = -\dfrac{\mu}{r_3}(r\dot{r}\vec{r} - \vec{r}^2\vec{r}^{\bullet}) = -\mu\left(\dfrac{\dot{r}}{r^2}\vec{r} - \dfrac{1}{r}\vec{r}^{\bullet}\right) = \mu\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\vec{r}}{r}\right)$
    - Note $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
    - $r\dot{r} = \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}r^2 + = \dfrac{2}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \cdot \vec{r}) = \dfrac{1}{2}(\vec{r}^{\bullet} \cdot \vec{r} + \vec{r} \cdot \vec{r}^{\bullet}) = \vec{r} \cdot \vec{r}^{\bullet}$

- $\underset{\rightarrow}{r}^{\cdot} \times \underset{\rightarrow}{h} = \mu\left(\frac{\underset{\rightarrow}{r}}{r} + \underset{\rightarrow}{\varepsilon}\right)$
  - $\underset{\rightarrow}{r} \cdot \underset{\rightarrow}{r}^{\cdot} \times \underset{\rightarrow}{h} = \mu(r + \underset{\rightarrow}{r} \cdot \underset{\rightarrow}{\varepsilon}) = \mu(r + \|\underset{\rightarrow}{r}\|\|\underset{\rightarrow}{\varepsilon}\| \cos\theta) = \mu r(1 + \varepsilon\cos\theta)$
  - $\underset{\rightarrow}{h} \times (\underset{\rightarrow}{r} \times \underset{\rightarrow}{r}^{\cdot}) = \underset{\rightarrow}{h} \cdot \underset{\rightarrow}{h} = h^2$
  - $h^2 = \mu r(1 + \varepsilon\cos\theta)$ or $r = \frac{\frac{h^2}{\mu}}{1 + \varepsilon\cos\theta}$
- $r = \frac{\frac{h^2}{\mu}}{1 + \varepsilon\cos\theta}$ is the shape of the orbit, which describes a general conic section (ellipse, parabola, hyperbola)
  - ε is the *eccentricity* of the conic section, $\varepsilon = 0$ gives a circle, $0 < \varepsilon < 1$ gives an ellipse, $\varepsilon = 1$ gives a parabola, $1 < \varepsilon$ gives a hyperbola
  - This gives us Kepler's first law
  - But also, we know that orbits can also be parabolas or hyperbolas
  - If we took the energy $e = \frac{1}{2}v^2 - \frac{\mu}{r}$ we can express everything in terms of it
- If between time t and $t + \mathrm{d}t$ the mass travelled an angle $\mathrm{d}\theta$, then the area swept out is $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$
  - $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h$, but we know this is constant from earlier
  - This gives us Kepler's second law
- $t = \sqrt{\frac{a^3}{\mu}}(E - \varepsilon\sin E)$
  - The period is $T = 2\pi\sqrt{\frac{a^3}{\mu}}$
  - This is Kepler's third law