Lecture 10, Oct 10, 2023

D'Alembert's Principle

- Consider an inertially fixed point $O_{\mathfrak{I}}$, an inertially moving point O and a grammar of particles $\mathcal{P}_i$, with positions $\vec{r}_i$ relative to $O_{\mathfrak{I}}$ and $\vec{\rho}_i$ relative to O
- How can we accommodate Newton's second law in a noninertial frame?
- $\vec{r}_i = \vec{r}^{O\mathfrak{I}} + \vec{\rho}_i$
- The equations of motion are $m_i \ddot{\vec{r}}_i = \vec{f}_{i,ext} + \sum_j \vec{f}_i^j = m_i(\ddot{\vec{\rho}}_i + \ddot{\vec{r}}^{O\mathfrak{I}})$
  - Let $\vec{a}_O = \ddot{\vec{r}}^{O\mathfrak{I}}$ which is the acceleration of O with respect to $O_{\mathfrak{I}}$
  - $m_i \ddot{\vec{\rho}}_i = \vec{f}_{i,ext} + \sum_i \vec{f}_i^j - m_i \vec{a}_O$
  - Therefore $m \ddot{\vec{\rho}}_{\otimes} = \vec{f} - m\vec{a}_O$
  - Let $\vec{\pi}_i = m_i \dot{\vec{\rho}}_i$ be the momentum as seen in O
  - $\vec{\pi} = m \dot{\vec{\rho}}_{\otimes}$
  - Therefore $\dot{\vec{\pi}} = \vec{f} - m\vec{a}_O$
  - Notice that the rate of change momentum as observed in O is the total force, plus a reversed inertial force
- For the angular momentum: $\vec{\eta}_i = \vec{\rho}_i \times \vec{\pi}_i$
  - The total momentum is then $\vec{\eta}_0 = \sum_i m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$
  - $\dot{\vec{\eta}}_0 = \sum_i \vec{\rho}_i \times (m_i \ddot{\vec{\rho}}_i) = \sum_i \vec{\rho}_i \times \left(\vec{f}_{i,ext} + \sum_j \vec{f}_i^j - m_i \vec{a}_O\right)$
  - $\dot{\vec{\eta}}_0 = \vec{\tau}_O - \left(\sum_i m_i \vec{\rho}_i\right) \times \vec{a}_O = \vec{\tau}_O - m\vec{\rho}_{\otimes} \times \vec{a}_O = \vec{\tau}_O + \vec{\rho}_{\otimes} \times (-m\vec{a}_O)$

Definition

D'Alembert's principle: The classical laws of mechanics can be applied in a linearly accelerating frame if reversed inertial forces are applied to the center of mass of a grammar of particles.

The Rocket Problem

- Consider a rocket with mass m moving at velocity $\vec{v}$; if $\vec{f} = \dot{\vec{p}}$, then do we have $\vec{f} = \dot{m}\vec{v} + m\dot{\vec{v}}$, since the mass is changing for the rocket?
  - The problem is that we failed to consider the part of the mass that was ejected
- At time t, the rocket has mass m and velocity $\vec{v}$; at time $t + \mathrm{d}t$, the rocket has mass $m + \mathrm{d}m$ and velocity $\vec{v} + \mathrm{d}\vec{v}$, where $-\mathrm{d}m$ is the amount of mass that was ejected at a velocity $\vec{v} + \vec{v}_{ex}$
- $\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \implies \mathrm{d}\vec{p} = \vec{f}\,\mathrm{d}t = \vec{p}(t + \mathrm{d}t) - \vec{p}(t)$
  - $\vec{p}(t) = m\vec{v}$
  - $\vec{p}(t + \mathrm{d}t) = (m + \mathrm{d}m)(\vec{v} + \mathrm{d}\vec{v}) + (-\mathrm{d}m)(\vec{v} + \vec{v}_{ex})$
  - Under a first order approximation, this reduces to $\mathrm{d}\vec{p} = m\mathrm{d}\vec{v} - \vec{v}_{ex}\mathrm{d}m = \vec{f}\,\mathrm{d}t$
  - Therefore $m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \vec{v}_{ex}\frac{\mathrm{d}m}{\mathrm{d}t} = \vec{f} \implies m\dot{\vec{v}} = \vec{f} + \dot{m}\vec{v}_{ex}$
- Comparing this result to what we have before, the difference is that instead of $-\vec{v}_{ex}$, in our first incorrect result we had $\vec{v}$
- Example: consider an open-top railway car with mass m_0 and velocity v_0, which is collecting rain at a rate $\dot{m}$; what is the velocity of the car after a given time, where an amount of rain m has fallen?
  - Suppose there is an amount of water m in the car at time t, then $p(t) = (m_0 + m)v$ and $p(t + \mathrm{d}t) = (m_0 + m + \mathrm{d}m)(v + \mathrm{d}v)$

    - Since the rain is falling vertically, $p(t) = p(t + \mathrm{d}t) \implies (m_0 + m)v = (m_0 + m + \mathrm{d}m)(v + \mathrm{d}v)$
    - $v\,\mathrm{d}m + (m_0 + m)\mathrm{d}v = 0 \implies \frac{\mathrm{d}v}{v} = -\frac{\mathrm{d}m}{m_0 + m}$ with initial conditions $m = 0, v = v_0$ at $t = 0$
    - Integrating both sides: $\ln\frac{v}{v_0} = -\ln\frac{m_0 + m}{m_0} \implies v = v_0\frac{m}{m_0 + m} \implies (m_0 + m)v = m_0 v_0$
- Example: if the car is dripping so that the water level stays the same, what is the velocity now?
    - $p(t) = m_0 v$
    - $p(t + \mathrm{d}t) = m_0(v + \mathrm{d}v) + (v + \mathrm{d}v)\mathrm{d}m$ where $-\mathrm{d}m$ is the amount of water falling out
    - Again $p(t + \mathrm{d}t) - p(t) = 0 \implies m_0\,\mathrm{d}v + v\,\mathrm{d}m = 0 \implies \frac{\mathrm{d}v}{v} = -\frac{\mathrm{d}m}{m_0} \implies \ln\frac{v}{v_0} = -\frac{m}{m_0} \implies v = v_0 e^{-\frac{m}{m_0}}$