Lecture 1, Sep 7, 2023

- *Postulate of Space*: physical space is represented by a 3-dimensional Euclidean space
- *Particle*: a quantity of matter that occupies an infinitesimal volume
- *Postulate of Time*: there exists the dimension of time, same regardless of space, and a particle can only be in one position at one time; changes in position are continuous in space and time
- *Law of Inertia*: there exists a frame of reference in which an isolated particle moves in a straight line in any direction (note we don't yet say the motion is uniform)
  - Such frames where the Law of Inertia holds are *inertial reference frames*
  - *Mach's Principle*: a local inertial frame is determined by all the matter in the universe and its distribution
- An isolated particle in an inertial reference frame moves equal intervals of distance in equal intervals of time; this is how we will define a graduation of time
- *Law of Action and Reaction*: the accelerations of two isolated particles in an inertial reference frame are in mutually opposite directions and the ratio of their magnitudes is constant
  - This allows us to define *ratio of masses* m_i to m_j is such that $m_i a_i = -m_j a_j$, but not absolute mass directly
- *Postulate of the Transitivity of Mass*: for any 3 particles, $\mu_{ij}\mu_{jk}\mu_{ki} = 1$ where μ_{ij} is the ratio of the mass of particle i to j
- *Total Force*: the total force $\underrightarrow{f}$ on a particle of mass m is $\underrightarrow{f} = m\underrightarrow{a}$
  - The *individual force* comes only from a single agent
  - Newton's second law becomes a definition for force
  - *Law of Superposition of Forces*: the total force on a particle is the vector sum of the individual forces

Lecture 2, Sep 12, 2023

Frames of Reference

- In this course, we restrict our analysis to dextral (right-handed) orthonormal frames of reference/bases
- Any vector $\underrightarrow{v}$ can be expressed in any basis: $\underrightarrow{v} = v_1\underrightarrow{a}_1 + v_2\underrightarrow{a}_2 + v_3\underrightarrow{a}_3$
  - v_1, v_2, v_3 are the coordinates of $\underrightarrow{v}$ in the basis $\underrightarrow{a}$
- As an alternative notation, consider $\underrightarrow{v} = \underrightarrow{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v} = v_1\underrightarrow{a}_1 + v_2\underrightarrow{a}_2 + v_3\underrightarrow{a}_3$, where $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\underrightarrow{\boldsymbol{\mathcal{F}}} = \begin{bmatrix} \underrightarrow{a}_1 \\ \underrightarrow{a}_2 \\ \underrightarrow{a}_3 \end{bmatrix}$
  - $\underrightarrow{\boldsymbol{\mathcal{F}}}$ is a matrix of vectors, so we call it a *vectrix*
  - We can also write this as $\underrightarrow{v} = \boldsymbol{v}^T\underrightarrow{\boldsymbol{\mathcal{F}}}$ which gives the same result
- We can now define various operations using vectrix notation:
  - $\underrightarrow{u} \cdot \underrightarrow{v} = (\boldsymbol{u}^T\underrightarrow{\boldsymbol{\mathcal{F}}}) \cdot (\underrightarrow{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v}) = \boldsymbol{u}^T(\underrightarrow{\boldsymbol{\mathcal{F}}} \cdot \underrightarrow{\boldsymbol{\mathcal{F}}}^T)\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{1}\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v}$
    - Note that $\underrightarrow{\boldsymbol{\mathcal{F}}} \cdot \underrightarrow{\boldsymbol{\mathcal{F}}}^T = \begin{bmatrix} \underrightarrow{a}_1 \cdot \underrightarrow{a}_1 & \underrightarrow{a}_1 \cdot \underrightarrow{a}_2 & \underrightarrow{a}_1 \cdot \underrightarrow{a}_3 \\ \underrightarrow{a}_2 \cdot \underrightarrow{a}_1 & \underrightarrow{a}_2 \cdot \underrightarrow{a}_2 & \underrightarrow{a}_2 \cdot \underrightarrow{a}_3 \\ \underrightarrow{a}_3 \cdot \underrightarrow{a}_1 & \underrightarrow{a}_3 \cdot \underrightarrow{a}_2 & \underrightarrow{a}_3 \cdot \underrightarrow{a}_3 \end{bmatrix} = \boldsymbol{1}$ because the basis in $\underrightarrow{\boldsymbol{\mathcal{F}}}$ is orthonormal
    - This definition is consistent with our usual definition of an inner product
  - $\underrightarrow{u} \times \underrightarrow{v} = (\boldsymbol{u}^T\underrightarrow{\boldsymbol{\mathcal{F}}}) \times (\underrightarrow{\boldsymbol{\mathcal{F}}}^T\boldsymbol{v}) = \boldsymbol{u}^T(\underrightarrow{\boldsymbol{\mathcal{F}}} \times \underrightarrow{\boldsymbol{\mathcal{F}}}^T)\boldsymbol{v} = (u_2v_3 - u_3v_2)\underrightarrow{a}_1 + (u_3v_1 - u_1v_3)\underrightarrow{a}_2 + (u_1v_2 - u_2v_1)\underrightarrow{a}_3$
    - Note that $\underrightarrow{\boldsymbol{\mathcal{F}}} \times \underrightarrow{\boldsymbol{\mathcal{F}}}^T = \begin{bmatrix} \underrightarrow{a}_1 \times \underrightarrow{a}_1 & \underrightarrow{a}_1 \times \underrightarrow{a}_2 & \underrightarrow{a}_1 \times \underrightarrow{a}_3 \\ \underrightarrow{a}_2 \times \underrightarrow{a}_1 & \underrightarrow{a}_2 \times \underrightarrow{a}_2 & \underrightarrow{a}_2 \times \underrightarrow{a}_3 \\ \underrightarrow{a}_3 \times \underrightarrow{a}_1 & \underrightarrow{a}_3 \times \underrightarrow{a}_2 & \underrightarrow{a}_3 \times \underrightarrow{a}_3 \end{bmatrix} = \begin{bmatrix} 0 & \underrightarrow{a}_3 & -\underrightarrow{a}_2 \\ -\underrightarrow{a}_3 & 0 & \underrightarrow{a}_1 \\ \underrightarrow{a}_2 & -\underrightarrow{a}_1 & 0 \end{bmatrix}$ where we have assumed right-handedness
    - Also note that $\underrightarrow{u} \times \underrightarrow{v} = -\underrightarrow{v} \times \underrightarrow{u} \iff \boldsymbol{u}^\times\boldsymbol{v} = -\boldsymbol{v}^\times\boldsymbol{u}$, i.e. the cross product is anti-commutative
    - This definition is also consistent with our usual definition for cross product
    - Alternatively $\underrightarrow{u} \times \underrightarrow{v} = \underrightarrow{\boldsymbol{\mathcal{F}}}^T \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \underrightarrow{\boldsymbol{\mathcal{F}}}^T\boldsymbol{u}^\times\boldsymbol{v}$ where $\boldsymbol{u}^\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$
      - $\boldsymbol{u}^\times$ is also known as the skew-symmetric form of $\boldsymbol{u}$ since $(\boldsymbol{u}^\times)^T = -\boldsymbol{u}^\times$

        - Also note that as expected $\det(\boldsymbol{u}^\times) = 0$ (in fact the determinant of any odd-dimensioned skew-symmetric matrix is zero)
  - $\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} \times \underset{\rightarrow}{w} = \boldsymbol{u}^T \underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \cdot (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T \boldsymbol{v}^\times \boldsymbol{w}) = \boldsymbol{u}^T \boldsymbol{v}^\times \boldsymbol{w}$
    * This is called the triple product and represents the volume of a parallelepiped formed by the 3 vectors
  - $\underset{\rightarrow}{u} \times (\underset{\rightarrow}{v} \times \underset{\rightarrow}{w}) = \boldsymbol{u}^T \underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \times (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T \boldsymbol{v}^\times \boldsymbol{w}) = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T \boldsymbol{u}^\times \boldsymbol{v}^\times \boldsymbol{w}$
  - $(\underset{\rightarrow}{u} \times \underset{\rightarrow}{v}) \times \underset{\rightarrow}{w} = (\boldsymbol{u}^\times \boldsymbol{v})^T \underset{\rightarrow}{\boldsymbol{\mathcal{F}}} \times \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T \boldsymbol{w} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}^T (\boldsymbol{u}^\times \boldsymbol{v})^\times \boldsymbol{w}$
    * Notice that this is in general not the same as the result above, so the cross product is not associative
- Consider two frames a and b, then $\underset{\rightarrow}{v} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b$; how do we relate $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$?
  - $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b \implies \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b \implies \boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b = \boldsymbol{C}_{ab} \boldsymbol{v}_b$
  - $\boldsymbol{C}_{ab}$ is our transformation matrix from b to a
  - Expanded: $\boldsymbol{C}_{ab} = \begin{bmatrix} \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{a}_1 \cdot \underset{\rightarrow}{b}_3 \\ \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{a}_2 \cdot \underset{\rightarrow}{b}_3 \\ \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{a}_3 \cdot \underset{\rightarrow}{b}_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta_{11}) & \cos(\theta_{12}) & \cos(\theta_{13}) \\ \cos(\theta_{21}) & \cos(\theta_{22}) & \cos(\theta_{23}) \\ \cos(\theta_{31}) & \cos(\theta_{32}) & \cos(\theta_{33}) \end{bmatrix}$
  - Note some properties of $\boldsymbol{C}_{ab}$:
    * $\boldsymbol{C}_{ab}\boldsymbol{C}_{ba} = \boldsymbol{C}_{aa} = \boldsymbol{1}$, so $\boldsymbol{C}_{ab} = \boldsymbol{C}_{ba}^{-1}$
    * $\boldsymbol{C}_{ab} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T = (\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T)^T = \boldsymbol{C}_{ba}$
    * Therefore $\boldsymbol{C}_{ab}^{-1} = \boldsymbol{C}_{ab}^T$
    * Since $\boldsymbol{C}^T\boldsymbol{C} = \boldsymbol{1}$ for all rotation matrices, $\det \boldsymbol{C}^2 = 1$ so $\det \boldsymbol{C} = 1$
      - Note the determinant of $\boldsymbol{C}$ can be negative when going from a left-handed to a right-handed frame and vice versa but we will not consider these in this course

Lecture 3, Sep 14, 2023

Rotation Matrices

- Recall that for $\underset{\rightarrow}{v} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{v}_a = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b$ we have $\boldsymbol{v}_a = \boldsymbol{C}_{ab}\boldsymbol{v}_b$, so substituting in the relation gives us $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{C}_{ab} \boldsymbol{v}_b = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b$, which gives us $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{C}_{ab} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T$, or $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a = \boldsymbol{C}_{ab} \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b$
  - Dot product in different frames: $\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} = \boldsymbol{u}_a^T \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{v}_b = \boldsymbol{u}_a^T \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a \cdot \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^T \boldsymbol{C}_{ab} \boldsymbol{v}_b = \boldsymbol{u}_a^T \boldsymbol{C}_{ab} \boldsymbol{v}_b$
- *Principal rotations* are rotations about one of the basis vectors
  - Example: rotation about $\underset{\rightarrow}{b}_1$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \cos\left(\frac{\pi}{2} - \theta\right) \\ 0 & \cos\left(\frac{\pi}{2} + \theta\right) & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \boldsymbol{C}_1(\theta)$
  - We use the notation $\boldsymbol{C}_n(\theta)$ to represent a principal rotation matrix about the nth basis by θ
  - $\boldsymbol{C}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$
  - $\boldsymbol{C}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- We can form compound rotation matrices by multiplying principal rotation matrices, e.g. $\boldsymbol{C}_{ac} = \boldsymbol{C}_{ab}\boldsymbol{C}_{bc}$
  - Notice that the subscripts match and kind of "cancel out"

Example: Universal Joint

- Example: for a universal joint as pictured, if we turn the front shaft by an angle θ, what is the angle ϕ that the rear shaft turns?
  - The front and rear shafts are at an angle α
  - Reference frames a and b are fixed to the two ends of the shaft respectively (and do not rotate); the frames c and d rotate with the shafts; the frame e is attached to the spider
    * $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b = \boldsymbol{C}_{ba} \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a = \boldsymbol{C}_1(\alpha) \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a$
    * $\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_c = \boldsymbol{C}_{ca} \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a = \boldsymbol{C}_2(\theta) \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a$

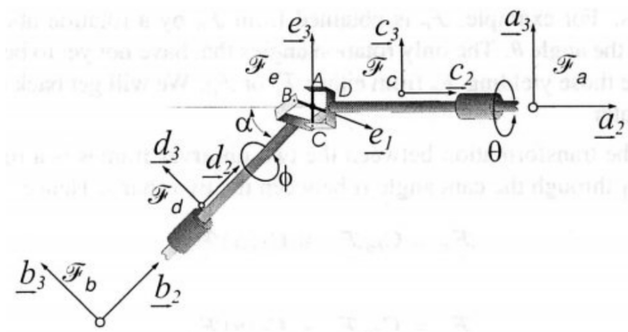

Figure 1: A universal joint with reference frames marked.

    * $\underrightarrow{\mathcal{F}}_d = C_{bd}\underrightarrow{\mathcal{F}}_b = \boldsymbol{C}_2(\phi)\underrightarrow{\mathcal{F}}_b$
  - Since one arm of the spider is fixed to the front shaft and the other is fixed to the rear shaft, the two arms are parallel to the two shafts respectively
    * $\underrightarrow{e}_1 = \underrightarrow{d}_1$
    * $\underrightarrow{e}_3 = \underrightarrow{c}_3$
  - With the connection between the reference frames now established, we can relate everything to frame a
    * $\underrightarrow{e}_1 = \underrightarrow{d}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \underrightarrow{\mathcal{F}}_d = \boldsymbol{1}_1^T \underrightarrow{\mathcal{F}}_d = \boldsymbol{1}_1^T \boldsymbol{C}_2(\phi)\underrightarrow{\mathcal{F}}_b = \boldsymbol{1}_1^T \boldsymbol{C}_2(\phi)\boldsymbol{C}_1(\alpha)\underrightarrow{\mathcal{F}}_a$
    * $\underrightarrow{\mathcal{F}}_e = \boldsymbol{C}_{ec}\underrightarrow{\mathcal{F}}_c = \boldsymbol{C}_3(\psi)\underrightarrow{\mathcal{F}}_c$
      - Since b and e share a common axis $\underrightarrow{e}_3 = \underrightarrow{c}_3$ they must be related by a principal rotation through axis 3; we don't know what the angle is, but we will declare it ψ and hope that it cancels
    * $\underrightarrow{e}_1 = \boldsymbol{1}_1^T \underrightarrow{\mathcal{F}}_e = \boldsymbol{1}_1^T \boldsymbol{C}_3(\psi)\underrightarrow{\mathcal{F}}_c = \boldsymbol{1}_1^T \boldsymbol{C}_3(\psi)\boldsymbol{C}_2(\theta)\underrightarrow{\mathcal{F}}_a$
  - Now we can equate $\underrightarrow{e}_1$ and solve for the relations: $\boldsymbol{1}_1^T \boldsymbol{C}_2(\phi)\boldsymbol{C}_1(\alpha) = \boldsymbol{1}_1^T \boldsymbol{C}_3(\psi)\boldsymbol{C}_2(\theta)$
  - The matrix equation can be expanded to obtain:
    * $\cos\phi = \cos\theta\cos\psi$
    * $\sin\alpha\sin\phi = \sin\psi$
    * $\cos\alpha\sin\phi = \sin\theta\cos\psi$
  - We can divide equation 3 by equation 1 to get $\cos\alpha\tan\phi = \tan\theta$ for our final result

Euler's Theorem

Theorem

Euler's Theorem: Any arbitrary rotation or sequence of rotations can be described by a single rotation about some axis.

- Proof:
  - Consider some arbitrary rotation matrix $\boldsymbol{C}$ and the eigenvalue problem $\boldsymbol{C}\boldsymbol{e} = \lambda\boldsymbol{e}$
  - Take the Hermitian (transpose and conjugate) so $\boldsymbol{e}^H\boldsymbol{C}^H = \bar{\lambda}\boldsymbol{e}^H$
    * Since $\boldsymbol{C}$ is real, $\boldsymbol{C}^H = \boldsymbol{C}^T$, but the eigenvalues and eigenvectors could be complex
  - Multiply by the Hermitian again on both sides: $\boldsymbol{e}^H\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{e} = \lambda\bar{\lambda}\boldsymbol{e}^H\boldsymbol{e} \implies \boldsymbol{e}^H\boldsymbol{e} = \lambda\bar{\lambda}\boldsymbol{e}^H\boldsymbol{e}$
  - $(\lambda\bar{\lambda} - 1)\boldsymbol{e}^H\boldsymbol{e} = 0$ and since the eigenvector is nonzero, $\lambda\bar{\lambda} = |\lambda|^2 = 1$
    * This means $\lambda = \pm 1$ or $e^{\pm j\phi}$ (complex conjugate pairs)
  - Since $\boldsymbol{C}$ is a 3 by 3 matrix, there are 3 eigenvalues,
    * Since the determinant of $\boldsymbol{C}$ is the product of its eigenvalues, we know its eigenvalues are all positive, otherwise the determinant could be negative
    * This necessitates that the eigenvalues are $\lambda = 1, e^{\pm j\phi}$

- Let the eigenvector with eigenvalue 1 be $\boldsymbol{a}$; then $\boldsymbol{C}\boldsymbol{a} = \boldsymbol{a}$, i.e. $\boldsymbol{a}$ is invariant under the rotation
    - This means it must be the axis of rotation!
  - Thus any rotation matrix has some axis of rotation, so any rotation or sequence of rotations corresponds to a rotation about some axis
  - It turns out that also ϕ is the angle of rotation (but this will not be proven right now)
- We can write any rotation in the form $\boldsymbol{C} = \cos\phi\boldsymbol{1} + (1 - \cos\phi)\boldsymbol{a}\boldsymbol{a}^T - \sin\phi\boldsymbol{a}^\times$ where $\boldsymbol{a}$ is the axis and ϕ is the angle

Lecture 4, Sep 19, 2023

Solving the Sundial

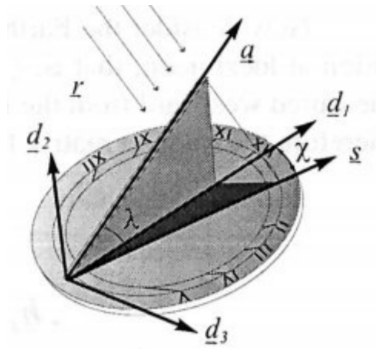

Figure 2: A sundial.

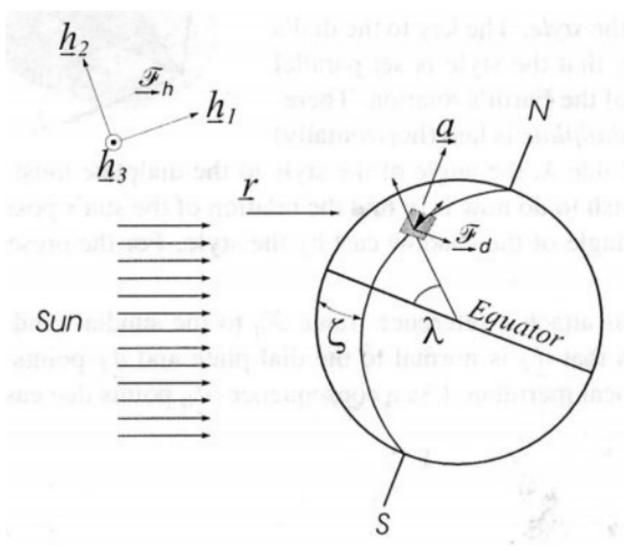

Figure 3: The geometry of the sundial.

- A sundial's upper edge, the *style*, is made parallel to the Earth's axis of rotation, so the sun casts a shadow on the plane of the sundial; how should we draw the markings on the sundial to indicate time?
- We will establish a coordinate system with $\underrightarrow{d}_1, \underrightarrow{d}_3$ being in the plane of the sundial
  - The shadow cast by the style is $\underrightarrow{s} = s_1\underrightarrow{d}_1 + s_2\underrightarrow{d}_3$
  - The angle the shadow makes with $\underrightarrow{d}_1$ is χ, so $\tan\chi = \dfrac{s_3}{s_1}$
  - The shadow $\underrightarrow{s}$, style $\underrightarrow{a}$, and the sun's rays $\underrightarrow{r}$ are in the same plane, so we have $\underrightarrow{s} = c\underrightarrow{a} + \underrightarrow{r}$ (since we don't care about the magnitude, we only need 1 coefficient)

- Therefore $s_1\underrightarrow{d}_1 + s_3\underrightarrow{d}_3 = c\underrightarrow{a} + \underrightarrow{r} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \begin{bmatrix} s_1 \\ 0 \\ s_3 \end{bmatrix} = s_1\mathbf{1}_1 + s_3\mathbf{1}_3$

- We define another frame, the heliocentric frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_h$, which is aligned with $\underrightarrow{\boldsymbol{\mathcal{F}}}_d$ at noon (when the rotation angle of the earth, $\zeta = 0$)
  - Consider the coordinate system formed by $\underrightarrow{a}, \underrightarrow{h}_3 \times \underrightarrow{a}$ and $\underrightarrow{h}_3$
  - The angle α between the sun's rays $\underrightarrow{r}$ and the style/Earth axis $\underrightarrow{a}$ is seasonally dependent
    * We can get break down $\underrightarrow{r}$ into $\underrightarrow{a}$ and a perpendicular component: $\underrightarrow{r} = \cos\alpha\underrightarrow{a} - \sin\alpha\underrightarrow{h}_3 \times \underrightarrow{a}$
    * $\underrightarrow{a} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \boldsymbol{a} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}$ where λ is the latitude
  - $\underrightarrow{r} = \underrightarrow{\boldsymbol{\mathcal{F}}}_h^T \boldsymbol{r}_h = \underrightarrow{\boldsymbol{\mathcal{F}}}_h^T \left(\cos\alpha\boldsymbol{a} - \sin\alpha\mathbf{1}_3^\times\boldsymbol{a}\right)$
- When the Earth rotates, the frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_d$ rotates about the axis $\underrightarrow{a}$ by ζ, so $\boldsymbol{C}_{dh} = \cos\zeta\mathbf{1} + (1-\cos\zeta)\boldsymbol{a}\boldsymbol{a}^T - \sin\zeta\boldsymbol{a}^\times$
- Now we have $\underrightarrow{a}$ and $\underrightarrow{r}$ in frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_d$, we can use the equation before: $c\underrightarrow{a} + \underrightarrow{r} = \underrightarrow{\boldsymbol{\mathcal{F}}}_d^T \begin{bmatrix} s_1 \\ 0 \\ s_3 \end{bmatrix}$
  - This gives us 3 equations, when solved we get $\tan\chi = \dfrac{s_3}{s_1} = \sin\lambda\tan\zeta$
- In general, we want to express everything in the same frame to solve a problem; some vectors are more easily expressed in certain frames than others

Rotation Representations

- We can represent any rotation with a rotation matrix, but it is overspecified since there are 9 components
- Using Euler's theorem, we can completely specify a rotation by the axis-angle pair $(\boldsymbol{a}, \phi)$, which has 3 components only (since $\boldsymbol{a}$ is normalized)
- We can also perform a series of 3 principal axis rotations; these are known as the Euler angles
  - Any sequence of 3 axes works, as long as you don't have a sequence of 2 consecutive identical rotations
  - e.g. $\boldsymbol{C} = \boldsymbol{C}_3(\theta_3)\boldsymbol{C}_1(\theta_2)\boldsymbol{C}_3(\theta_1)$ is a 3-1-3 set of rotations
    * The 3-1-3 set is the one that Euler used
  - In total there are $3 \times 2 \times 2 = 12$ different sets of rotations
  - Given $\boldsymbol{C}$, we can't always find a unique sequence of Euler angles that make up the rotation; this is referred to as a *singularity*
    * e.g. with the 3-1-3 set, if $\theta_2 = 0$ then we won't be able to distinguish θ_3 from θ_1
    * This is why in aerospace typically 1-2-3 or 3-2-1 is used; however any sequence has a singularity, it's just for some sequences the singularity occurs further from the reference point

Kinematics

- Kinematics is the geometry of motion, with no regard for the laws of nature
- Rates of change like $\underrightarrow{v}$ depend on the frame of reference that they are viewed from; when we take a derivative we must note the reference frame

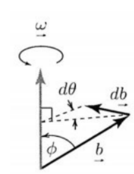

Figure 4: Vector of a fixed length in rotation.

- Consider a vector $\underrightarrow{b}$ of fixed length rotating about a fixed axis $\underrightarrow{a}$ with some rate $\dot{\theta}$

- Since $\underset{\rightarrow}{b}$ is constant length, $\mathrm{d}\underset{\rightarrow}{b}$ is normal to $\underset{\rightarrow}{b}$ and $\underset{\rightarrow}{a}$, so $\mathrm{d}\underset{\rightarrow}{b} \propto \underset{\rightarrow}{a} \times \underset{\rightarrow}{b}$
  - Suppose that in time $\mathrm{d}t$, $\underset{\rightarrow}{b}$ rotated $\mathrm{d}\theta$; then $\|\mathrm{d}\underset{\rightarrow}{b}\| = \|\underset{\rightarrow}{b}\| \sin\phi \mathrm{d}\theta$
  - If we let $\underset{\rightarrow}{\omega} = \underset{\rightarrow}{a}\dot{\theta}$, then $\mathrm{d}\theta = \|\underset{\rightarrow}{\omega}\|\,\mathrm{d}t$ and so $\|\mathrm{d}\underset{\rightarrow}{b}\| = \|\underset{\rightarrow}{\omega} \times \underset{\rightarrow}{b}\|\,\mathrm{d}t$
  - Therefore we have $\mathrm{d}\underset{\rightarrow}{b} = \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{b}\,\mathrm{d}t$ and so $\left.\frac{\mathrm{d}\underset{\rightarrow}{b}}{\mathrm{d}t}\right|_{\underset{\rightarrow}{\mathcal{F}}_a} = \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{b}$
- We will use the notation that $\underset{\rightarrow}{b}_1^{\bullet} = \left.\frac{\mathrm{d}\underset{\rightarrow}{b}_1}{\mathrm{d}t}\right|_{\underset{\rightarrow}{\mathcal{F}}_a}$ and $\underset{\rightarrow}{b}_1^{\circ} = \left.\frac{\mathrm{d}\underset{\rightarrow}{b}_1}{\mathrm{d}t}\right|_{\underset{\rightarrow}{\mathcal{F}}_b}$
- Since the axes of reference frames have constant length, we can use this for all 3 axis vectors
  - We denote $\underset{\rightarrow}{\mathcal{F}}_b^{T\bullet} = \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\mathcal{F}}_b^T = \begin{bmatrix}\underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{b}_1 & \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{b}_2 & \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{b}_3\end{bmatrix}$
    - The $\underset{\rightarrow}{\omega}^{ba}$ is the angular velocity of $\underset{\rightarrow}{\mathcal{F}}_b$ with respect to $\underset{\rightarrow}{\mathcal{F}}_a$
  - $\underset{\rightarrow}{\mathcal{F}}_b^{T\bullet} = \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\mathcal{F}}_b^T = \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{\omega}^{ba\times} \implies \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T\bullet} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{\omega}^{ba\times} \implies \boldsymbol{\omega}^{ba\times} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T\bullet}$
- $\boldsymbol{\omega}^{ba\times} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T\bullet}$ becomes our definition for angular velocity of frame b with respect to a in general

Lecture 5, Sep 21, 2023

Transport Equations

- We want to know how $\underset{\rightarrow}{v}^{\bullet}$ is related to $\underset{\rightarrow}{v}^{\circ}$
  - $\underset{\rightarrow}{v} = \underset{\rightarrow}{\mathcal{F}}_a^T \boldsymbol{v}_a = \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{v}_b$
  - $\underset{\rightarrow}{v}^{\bullet} = \underset{\rightarrow}{\mathcal{F}}_a^{T\bullet} \boldsymbol{v}_a + \underset{\rightarrow}{\mathcal{F}}_a^T \boldsymbol{v}_a^{\bullet} = \underset{\rightarrow}{\mathcal{F}}_a^T \boldsymbol{v}_b^{\bullet}$
  - $\underset{\rightarrow}{v}^{\circ} = \underset{\rightarrow}{\mathcal{F}}_b^{T\circ} \boldsymbol{v}_b + \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{v}_b^{\circ} = \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{v}_b^{\circ}$
  - $\underset{\rightarrow}{v}^{\bullet} = \underset{\rightarrow}{\mathcal{F}}_b^T \dot{\boldsymbol{v}}_b + \underset{\rightarrow}{\mathcal{F}}_b^{T\bullet} \boldsymbol{v}_b = \underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{v}_b = \underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}$
    - Note that when we have as scalar $\boldsymbol{v}_b$, we put the dot over instead of to the right, because for the scalar reference frames does not matter
- Therefore $\underset{\rightarrow}{v}^{\bullet} = \underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}$, which is known as the *transport equation* for velocity
- For acceleration, $\underset{\rightarrow}{v}^{\bullet\bullet} = (\underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v})^{\bullet}$

$$\begin{aligned}
&= (\underset{\rightarrow}{v}^{\circ})^{\bullet} \underset{\rightarrow}{\omega}^{ba\bullet} \times \underset{\rightarrow}{v} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}^{\bullet} \\
&= \underset{\rightarrow}{v}^{\circ\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}^{\circ} (\underset{\rightarrow}{\omega}^{ba\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\omega}^{ba}) \times \underset{\rightarrow}{v} + \underset{\rightarrow}{\omega}^{ba} \times (\underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}) \\
&= \underset{\rightarrow}{v}^{\circ\circ} + 2\underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{ba\circ} \times \underset{\rightarrow}{v} + \underset{\rightarrow}{\omega}^{ba} \times (\underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v})
\end{aligned}$$

  - Notice that $\underset{\rightarrow}{\omega}^{ba\bullet} = \underset{\rightarrow}{\omega}^{ba\circ}$
  - If we interpret $\underset{\rightarrow}{v}$ as a position vector, then $2\underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{v}^{\circ}$ would be the Coriolis acceleration, $\underset{\rightarrow}{\omega}^{ba} \times (\underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{r})$ would be the centripetal acceleration and $\underset{\rightarrow}{\omega}^{ba\circ} \times \underset{\rightarrow}{r}$ would be the tangential acceleration (or Euler acceleration)
- In coordinate form, the velocity transport equation is $\dot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\dot{\boldsymbol{v}}_b + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{v}_b)$
- For acceleration this is $\ddot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\ddot{\boldsymbol{v}}_b + 2\boldsymbol{\omega}_b^{ba\times} \dot{\boldsymbol{v}}_b + \dot{\boldsymbol{\omega}}_b^{ba\times} \boldsymbol{v} + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{\omega}_b^{ba\times} \boldsymbol{v}_b)$
- How do rotation matrices change with rotating reference frames?
  - $\underset{\rightarrow}{\mathcal{F}}_b^{T\bullet} = \underset{\rightarrow}{\mathcal{F}}_a^T \dot{\boldsymbol{C}}_{ab}$ since $\underset{\rightarrow}{\mathcal{F}}_a^{T\bullet} = \boldsymbol{0}$
  - $\boldsymbol{\omega}_b^{ba\times} = \underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^{T\bullet} = \boldsymbol{C}_{ba} \underset{\rightarrow}{\mathcal{F}}_a \cdot \underset{\rightarrow}{\mathcal{F}}_a^T \dot{\boldsymbol{C}}_{ab} = \boldsymbol{C}_{ba} \dot{\boldsymbol{C}}_{ab}$
  - If we transpose both sides we get $-\boldsymbol{\omega}_b^{ba\times} = \dot{\boldsymbol{C}}_{ba} \boldsymbol{C}_{ab}$
  - Multiplying by $\boldsymbol{C}_{ba}$ and rearranging, we get $\dot{\boldsymbol{C}}_{ba} + \boldsymbol{\omega}_b^{ba\times} \boldsymbol{C}_{ba} = \boldsymbol{0}$
    - We have found a differential equation for the rotation matrix that describes its evolution
    - This is known as *Poisson's kinematical equation*
- Consider now 3 reference frames $\underset{\rightarrow}{\mathcal{F}}_a, \underset{\rightarrow}{\mathcal{F}}_b, \underset{\rightarrow}{\mathcal{F}}_c$; what is the relationship among $\underset{\rightarrow}{\omega}^{ba}, \underset{\rightarrow}{\omega}^{cb}$ and $\underset{\rightarrow}{\omega}^{ca}$?
  - $\dot{\boldsymbol{C}}_{ca} = \dot{\boldsymbol{C}}_{cb} \boldsymbol{C}_{ba} + \boldsymbol{C}_{cb} \dot{\boldsymbol{C}}_{ba}$

- $\boldsymbol{\omega}_c^{ca\times} = -\dot{\boldsymbol{C}}_{ca}\boldsymbol{C}_{ac}$
  $= -\dot{\boldsymbol{C}}_{cb}\boldsymbol{C}_{ba}\boldsymbol{C}_{ab}\boldsymbol{C}_{bc} - \boldsymbol{C}_{cb}\dot{\boldsymbol{C}}_{ba}\boldsymbol{C}_{ab}\boldsymbol{C}_{bc}$
  $= -\dot{\boldsymbol{C}}_{cb}\boldsymbol{C}_{bc} - \boldsymbol{C}_{cb}\dot{\boldsymbol{C}}_{ba}\boldsymbol{C}_{ab}\boldsymbol{C}_{bc}$
  $= \boldsymbol{\omega}_c^{cb\times} + \boldsymbol{C}_{cb}\boldsymbol{\omega}_b^{ba\times}\boldsymbol{C}_{bc}$
  $= \boldsymbol{\omega}_c^{cb\times} + (\boldsymbol{C}_{cb}\boldsymbol{\omega}_b^{ba})^\times$
  $= (\boldsymbol{\omega}_c^{cb} + \boldsymbol{C}_{cb}\boldsymbol{\omega}_b^{ba})^\times$
- Therefore $\boldsymbol{\omega}_c^{ca} = \boldsymbol{\omega}_c^{cb} + \boldsymbol{C}_{cb}\boldsymbol{\omega}_b^{ba}$
- If we multiply both sides by $\underrightarrow{\boldsymbol{\mathcal{F}}}_c^T$, we see that $\underrightarrow{\omega}^{ca} = \underrightarrow{\omega}^{cb} + \underrightarrow{\omega}^{ba}$

Important

While angular velocities can be added directly as $\underrightarrow{\omega}^{ca} = \underrightarrow{\omega}^{cb} + \underrightarrow{\omega}^{ba}$, angular accelerations cannot!

Summary

The transport equations relate velocity and acceleration as measured in one frame to how they're measured in another frame:

- $\underrightarrow{v}^{\bullet} = \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}$
- $\dot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\dot{\boldsymbol{v}}_b + \boldsymbol{\omega}_b^{ba\times}\boldsymbol{v}_b)$
- $\underrightarrow{v}^{\bullet\bullet} = \underrightarrow{v}^{\circ\circ} + 2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{\circ} + \underrightarrow{\omega}^{ba\circ} \times \underrightarrow{v} + \underrightarrow{\omega}^{ba} \times (\underrightarrow{\omega}^{ba} \times \underrightarrow{v})$
- $\ddot{\boldsymbol{v}}_a = \boldsymbol{C}_{ab}(\ddot{\boldsymbol{v}}_b + 2\boldsymbol{\omega}_b^{ba\times}\dot{\boldsymbol{v}}_b + \dot{\boldsymbol{\omega}}_b^{ba\times}\boldsymbol{v} + \boldsymbol{\omega}_b^{ba\times}\boldsymbol{\omega}_b^{ba\times}\boldsymbol{v}_b)$

Rotation Representations Revisited

- Given an axis-angle representation $\boldsymbol{a}, \phi$, the Euler parameters are $\eta = \cos\frac{1}{2}\phi$ and $\boldsymbol{\varepsilon} = \boldsymbol{a}\sin\frac{1}{2}\phi$
  - η and $\boldsymbol{\varepsilon}$ are not independent, because we stipulate that $\eta^2 + \boldsymbol{\varepsilon}^T\varepsilon = 1$
  - Euler parameters don't have a singularity
  - These are also known as quaternions
- Consider a 1-2-3 set of Euler angles, so $\boldsymbol{C} = \boldsymbol{C}_3\boldsymbol{C}_2\boldsymbol{C}_1$
  - $\boldsymbol{\omega}^\times = -\dot{\boldsymbol{C}}\boldsymbol{C}^T$ (Note here $\boldsymbol{\omega} = \boldsymbol{\omega}_b^{ba}$)
  - $\dot{\boldsymbol{C}} = \boldsymbol{C}_3\boldsymbol{C}_2\dot{\boldsymbol{C}}_1 + \boldsymbol{C}_3\dot{\boldsymbol{C}}_2\boldsymbol{C}_1 + \dot{\boldsymbol{C}}_3\boldsymbol{C}_2\boldsymbol{C}_1$
  - $\boldsymbol{\omega}^\times = -\boldsymbol{C}_3\boldsymbol{C}_2\dot{\boldsymbol{C}}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \boldsymbol{C}_3\dot{\boldsymbol{C}}_2\boldsymbol{C}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \dot{\boldsymbol{C}}_3\boldsymbol{C}_2\boldsymbol{C}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T$
    $= -\boldsymbol{C}_3\boldsymbol{C}_2\dot{\boldsymbol{C}}_1\boldsymbol{C}_1^T\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \boldsymbol{C}_3\dot{\boldsymbol{C}}_2\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \dot{\boldsymbol{C}}_3\boldsymbol{C}_3^T$
    $= \boldsymbol{C}_3\boldsymbol{C}_2\boldsymbol{1}_1^\times\dot{\theta}_1\boldsymbol{C}_2^T\boldsymbol{C}_3^T - \boldsymbol{C}_3\boldsymbol{1}_2^\times\dot{\theta}_2\boldsymbol{C}_3^T - \boldsymbol{1}_3^\times\dot{\theta}_3$
    $= (\boldsymbol{C}_3\boldsymbol{C}_2\boldsymbol{1}_1\dot{\theta}_1)^\times + (\boldsymbol{C}_3\boldsymbol{1}_2\dot{\theta}_2)^\times + \boldsymbol{1}_3^\times\dot{\theta}_3$
    - Note that $\dot{\boldsymbol{C}}_1\boldsymbol{C}_1^T$ is the angular velocity of a rotation about the first axis, so it is equal to $\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}^\times = \boldsymbol{1}_1^\times\dot{\theta}_1$
  - This means $\boldsymbol{\omega} = \boldsymbol{C}_3\boldsymbol{C}_2\boldsymbol{1}_1\dot{\theta}_1 + \boldsymbol{C}_3\boldsymbol{1}_2\dot{\theta}_2 + \boldsymbol{1}_3\dot{\theta}_3 = \begin{bmatrix}\boldsymbol{C}_3\boldsymbol{C}_2\boldsymbol{1}_1 & \boldsymbol{C}_3\boldsymbol{1}_2 & \boldsymbol{1}_3\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\\\dot{\theta}_3\end{bmatrix} = \boldsymbol{S}(\theta_2, \theta_3)\dot{\boldsymbol{\theta}}$
    - This is the kinematic equation for a 1-2-3 set of Euler angles (page 89 lists the $\boldsymbol{S}$ matrices for other combinations)
  - At a singularity, $\boldsymbol{S}$ becomes singular (hence the name singularity) – given $\boldsymbol{\omega}$, at a singularity, we cannot find $\dot{\boldsymbol{\theta}}$
- Given any rotation representation, we can write a kinematic equation for it

Lecture 6, Sep 26, 2023

Example Problem: Square-Dancing Ants

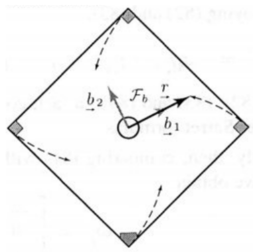

Figure 5: Example problem diagram.

- Consider 4 ants on the corners of a square with sides a; each ant directly walks towards the ant in front of it, so overall the ants all spiral inward; when the ants meet in the center, how far will each have walked?
- At any given time all the ants form a square provided their speeds are the same
- Let the speed of each ant be v, so that the path length s being walked by the ants at any given time is related to v as $v = \dot{s} = \dfrac{\mathrm{d}s}{\mathrm{d}t}$
- Construct our reference frame so that $\underset{\rightarrow}{b}_1$ and $\underset{\rightarrow}{b}_2$ point from the center of the square to two ants; $\underset{\rightarrow}{b}_3$ then points out of the page
  - This reference frame rotates since the ants move
- Let $\underset{\rightarrow}{\rho}$ be the position of one ant, so $\underset{\rightarrow}{\rho} = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} \rho \\ 0 \\ 0 \end{bmatrix}$ since $\underset{\rightarrow}{b}_1$ directly points towards the ant
- The velocity of the ant is $\underset{\rightarrow}{v} = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} -\dfrac{v}{\sqrt{2}} \\ \dfrac{v}{\sqrt{2}} \\ 0 \end{bmatrix} = \underset{\rightarrow}{\rho}^{\bullet} = \underset{\rightarrow}{\rho}^{\circ} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{\rho} = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} \dot{\rho} \\ 0 \\ 0 \end{bmatrix} + \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}^{\times} \begin{bmatrix} \rho \\ 0 \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} \dot{\rho} \\ \omega\rho \\ 0 \end{bmatrix}$
  - Therefore $-\dfrac{v}{\sqrt{2}} = -\dfrac{\dot{s}}{2} = \dot{\rho}$, $\dfrac{v}{\sqrt{2}} = \dfrac{\dot{s}}{\sqrt{2}} = \omega\rho$
  - Integrating the first equation: $\rho = \dfrac{1}{\sqrt{2}}(s_0 - s)$
  - We can determine s_0 by noting that at time 0, the distance $\rho = \dfrac{a}{\sqrt{2}}$ and $s = 0$, so $s_0 = a$
  - Now we can set $\rho = 0$ to solve for s: $0 = \dfrac{1}{\sqrt{2}}(a - s) \implies s = a$
- Therefore each ant travels precisely the same length as the sides of the square

Lecture 7, Sep 28, 2023

Newton's Second Law in Noninertial Frames

- Kinematics was the study of the geometry of motion without regard for the laws of nature; now we move on to dynamics, where we attempt to describe the laws of nature
- We know that the law of inertia does not hold in an accelerating or rotating reference frame; what about Newton's second law?
- $\underset{\rightarrow}{f} = m\underset{\rightarrow}{a} = m\underset{\rightarrow}{r}^{\bullet\bullet}$, with the derivative taken with respect to some inertial frame $\underset{\rightarrow}{\mathcal{F}}_I$

– In another frame $\underset{\rightarrow}{\mathcal{F}}_b$, $\underset{\rightarrow}{f} = m\underset{\rightarrow}{v}^{\cdot}$

$$\begin{aligned}
&= m(\underset{\rightarrow}{v}^{\circ} + \underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{v}) \\
&= m\underset{\rightarrow}{r}^{\cdot\cdot} \\
&= m(\underset{\rightarrow}{r}^{\circ\circ} + 2\underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{r}^{\circ} + \underset{\rightarrow}{\omega}^{bI\circ} \times \underset{\rightarrow}{r} + \underset{\rightarrow}{\omega}^{bI} \times (\underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{r}))
\end{aligned}$$

– So in $\underset{\rightarrow}{\mathcal{F}}_b$, $m\underset{\rightarrow}{r}^{\circ\circ} = \underset{\rightarrow}{f} - m\left(2\underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{r}^{\circ} - \underset{\rightarrow}{\omega}^{bI\circ} \times \underset{\rightarrow}{r} + \underset{\rightarrow}{\omega}^{bI} \times (\underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{r})\right)$

– We can see this broken down into the Coriolis, tangential, and centrifugal forces

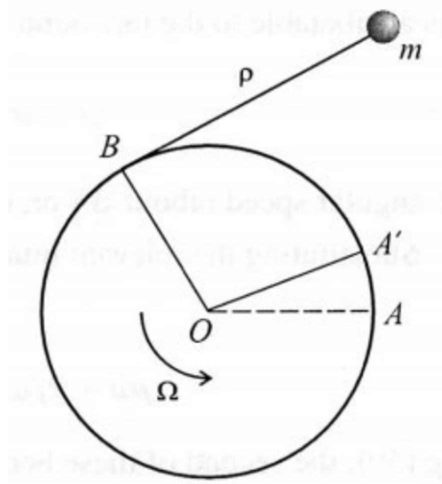

Figure 6: Diagram for the example problem.

- Example: Consider a spool with a bob attached to the end of the wire; if the spool is rotating in the opposite direction that the wire is being wound, the spool will actually unwind
  – Given that Ω is constant, what is $\rho(t)$ and $f_T(t)$?
  – Define our reference frames as $\underset{\rightarrow}{\mathcal{F}}_I$, the inertial frame, and $\underset{\rightarrow}{\mathcal{F}}_b$, a rotation reference frame with $\underset{\rightarrow}{b}_1$ parallel to the string at all times
  – This gives $\underset{\rightarrow}{r} = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} \rho \\ a \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{r}_b$ and $\underset{\rightarrow}{f}_T = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} -f_T \\ 0 \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{f}_b$
  – Since $\underset{\rightarrow}{\mathcal{F}}_b$ is not an inertial frame, we must use the equation of motion for a rotating frame that we derived above
  – $\underset{\rightarrow}{f}_T = m(\underset{\rightarrow}{r}^{\circ\circ} + 2\underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{r}^{\circ} + \underset{\rightarrow}{\omega}^{bI\circ} \times \underset{\rightarrow}{r} + \underset{\rightarrow}{\omega}^{bI} \times (\underset{\rightarrow}{\omega}^{bI} \times \underset{\rightarrow}{r}))$
  – It is most convenient to express all quantities in frame $\underset{\rightarrow}{\mathcal{F}}_b$:
    * $\underset{\rightarrow}{\omega}^{bI} = \underset{\rightarrow}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$
      - $\theta = \angle AOB - \dfrac{\pi}{2} = \angle AOA' + \angle A'OB - \dfrac{\pi}{2} = \Omega t + \dfrac{\rho}{a} - \dfrac{\pi}{2}$
        – Note the $\dfrac{\rho}{a}$ term comes from the fact that the arc length from B to A' is ρ
      - $\dot{\theta} = \Omega + \dfrac{\dot{\rho}}{a}$
    * $\dot{\boldsymbol{r}}_b = \begin{bmatrix} \dot{\rho} \\ 0 \\ 0 \end{bmatrix}, \ddot{\boldsymbol{r}}_b = \begin{bmatrix} \ddot{\rho} \\ 0 \\ 0 \end{bmatrix}$
  – If we substitute these quantities back in, we get $\begin{cases} -m\ddot{\rho} - m\rho\omega^2 = f_T \\ \rho\dot{\omega} + 2\dot{\rho}\omega - a\omega^2 = 0 \end{cases}$ where $\omega = \Omega + \dfrac{\dot{\rho}}{a}$
    * Solving the DE in the second equation, we get $\rho(t) = a\Omega t$
    * Substitute back in to get $f_T = 4ma\Omega^3 t$

- The idealized math says that the spool will keep unwinding, however in reality drag will eventually match the centrifugal force, causing the spool to no longer unwind

Lecture 8, Oct 3, 2023

Rotational Version of Newton's Laws

- Recall the momentum is $\underset{\to}{p} = m\dot{\underset{\to}{r}}$
- $\underset{\to}{h}_O = \underset{\to}{r} \times \underset{\to}{p}$ is the *angular momentum*, or moment of momentum
- $\dot{\underset{\to}{h}}_O = \dot{\underset{\to}{r}} \times \underset{\to}{p} + \underset{\to}{r} \times \dot{\underset{\to}{p}} = m\dot{\underset{\to}{r}} \times \dot{\underset{\to}{r}} + \underset{\to}{r} \times \underset{\to}{f} = \underset{\to}{r} \times \underset{\to}{f} = \underset{\to}{\tau}_O$ is the *torque*, or moment of force
  - Note that when we talk about moments such as angular momentum or torque, we need some reference point O
  - Here O is assumed to be inertially fixed; if it moves, then $\underset{\to}{\tau}_O = \dot{\underset{\to}{h}}_O + \underset{\to}{v}_O \times \underset{\to}{p}$ where $\underset{\to}{v}_O$ is the moment with respect to inertial space
- $\underset{\to}{f} = \dot{\underset{\to}{p}} = \overset{\circ}{\underset{\to}{p}} + \underset{\to}{\omega} \times \underset{\to}{p}$
  - We can think of this as the *translational equation of motion*
- $\underset{\to}{\tau}_O = \dot{\underset{\to}{h}}_O = \overset{\circ}{\underset{\to}{h}}_O + \underset{\to}{\omega} \times \underset{\to}{h}_O$
  - We can think of this as the *rotational equation of motion*
  - But this is not a law because it is derivable from the other laws and assumptions
- Impulse is defined as $\underset{\to}{\imath} = \int_{t_a}^{t_b} \underset{\to}{f}\,\mathrm{d}t = \underset{\to}{p}_b - \underset{\to}{p}_a$
- Rotational impulse is $\underset{\to}{\jmath}_O = \int_{t_a}^{t_b} \underset{\to}{\tau}_O\,\mathrm{d}t = \underset{\to}{h}_{O,b} - \underset{\to}{h}_{O,a}$

Work and Energy

- $$\begin{aligned} W &= \int_A^B \underset{\to}{f} \cdot \mathrm{d}\underset{\to}{r} \\ &= \int_A^B m\ddot{\underset{\to}{r}} \cdot \mathrm{d}\underset{\to}{r} \\ &= m\int \frac{1}{2}\mathrm{d}v^2 \\ &= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \\ &= T_B - T_A \end{aligned}$$
  - T_A, T_B are the *kinetic energy*; this is known as the principle of work and kinetic energy
  - Note $v^2 = \dot{\underset{\to}{r}} \cdot \dot{\underset{\to}{r}}$ so $\frac{\mathrm{d}v^2}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\underset{\to}{r}} \cdot \dot{\underset{\to}{r}}}{\mathrm{d}t} = 2\ddot{\underset{\to}{r}} \cdot \dot{\underset{\to}{r}} = 2\ddot{\underset{\to}{r}} \cdot \frac{\mathrm{d}\underset{\to}{r}}{\mathrm{d}t}$
  - Therefore $\mathrm{d}v^2 = 2\ddot{\underset{\to}{r}} \cdot \mathrm{d}\underset{\to}{r}$
- A force $\underset{\to}{f}$ is *conservative* iff $\int_{P_a} \underset{\to}{f} \cdot \mathrm{d}\underset{\to}{r} = \int_{P_b} \underset{\to}{f} \cdot \mathrm{d}\underset{\to}{r}$ for any two paths P_a, P_b that have the same start and end points
  - Equivalently, $\underset{\to}{\nabla} \times \underset{\to}{f} = \underset{\to}{0}$ (no curl) or $\underset{\to}{f} = -\underset{\to}{\nabla}V$ or $\oint \underset{\to}{f} \cdot \mathrm{d}\underset{\to}{r} = 0$
  - If $\underset{\to}{f} = -\underset{\to}{\nabla}V$, then $\underset{\to}{f} \cdot \mathrm{d}\underset{\to}{r} = -\underset{\to}{\nabla}V \cdot \mathrm{d}\underset{\to}{r} = -\frac{\partial V}{\partial x_1}\mathrm{d}x_1 - \frac{\partial V}{\partial x_2}\mathrm{d}x_2 - \frac{\partial V}{\partial x_3}\mathrm{d}x_3 = \mathrm{d}V$
  - $\int_A^B \underset{\to}{f} \cdot \mathrm{d}\underset{\to}{r} = -\int \mathrm{d}V = V_A - V_B$, regardless of the path taken from A to B
- If we combine the above with the principle of work and kinetic energy, we see $V_A - V_B = T_B - T_A \implies T_A + V_A = T_B + V_B$
  - This is the *conservation of (total) energy* – under a conservative force field $\underset{\to}{f}$, the sum of kinetic and potential energies, $T + V$, is conserved
    - V is the *potential energy*

* $T + V = E$ is the *total (mechanical) energy*

Lecture 9, Oct 5, 2023

A Grammar (System) of Particles

- Consider a system of particles, $\mathcal{P}_i, \mathcal{P}_j, \ldots$, with the reference points $O_{\mathfrak{I}}$, an inertially fixed reference point, ◕, the centre-of-mass reference point, and O, an arbitrary reference point
  - Each particle has positions $\underset{\to}{r}_i, \underset{\to}{r}_j$ relative to $O_{\mathfrak{I}}$, $\underset{\to}{s}_i, \underset{\to}{s}_j$ relative to ◕, masses m_i, m_j, external forces $\underset{\to}{f}_{i,ext}, \underset{\to}{f}_{j,ext}$ and forces between particles $\underset{\to}{f}_i^j, \underset{\to}{f}_j^i$
- We wish to extend the concepts of momentum and angular momentum to this grammar of particles
  - $\underset{\to}{p}_i = m_i \dot{\underset{\to}{r}}_i = m_i \underset{\to}{v}_i$
  - The total momentum is $\underset{\to}{p} = \sum_i \underset{\to}{p}_i = \sum_i m_i \dot{\underset{\to}{r}}_i$
  - $\underset{\to}{h}_i = m_i \underset{\to}{r}_i \times \dot{\underset{\to}{r}}_i = m_i \underset{\to}{r}_i \times \underset{\to}{v}_i = \underset{\to}{r}_i \times \underset{\to}{p}_i$
  - Total angular momentum is $\underset{\to}{h} = \sum_i \underset{\to}{h}_i = \sum_i m_i \underset{\to}{r}_i \times \dot{\underset{\to}{r}}_i$
- The centre of mass is $\underset{\to}{r}_{\text{◕}} = \sum_i \frac{m_i \underset{\to}{r}_i}{m}$ where $m = \sum_i m_i$ is the total mass
  - Therefore $m\underset{\to}{r}_{\text{◕}} = \sum_i m_i \underset{\to}{r}_i \implies m\dot{\underset{\to}{r}}_{\text{◕}} = \sum_i m_i \dot{\underset{\to}{r}}_i = \underset{\to}{p}$ which is the total momentum
  - We can therefore work with momentum as though all particles are concentrated at the center of mass
- $\underset{\to}{r}_i = \underset{\to}{r}_{\text{◕}} + \underset{\to}{s}_i \implies \underset{\to}{p} = \sum_i m_i(\dot{\underset{\to}{r}}_{\text{◕}} + \dot{\underset{\to}{s}}_i) = \sum_i m_i \dot{\underset{\to}{r}}_{\text{◕}} + \sum_i m_i \dot{\underset{\to}{s}}_i = m\dot{\underset{\to}{r}}_{\text{◕}} + \sum_i m_i \dot{\underset{\to}{s}}_i$
  - Therefore $\sum_i m_i \dot{\underset{\to}{s}}_i = \underset{\to}{0}$; for an observer at ◕, the total momentum is zero
- What about forces?
  - $m_i \ddot{\underset{\to}{r}}_i = \underset{\to}{f}_{i,ext} + \sum_j \underset{\to}{f}_i^j$ assuming that $\underset{\to}{f}_i^i = \underset{\to}{0}$
  - Summing over all forces, $\sum_i m_i \ddot{\underset{\to}{r}}_i = m\ddot{\underset{\to}{r}}_{\text{◕}} = \sum_i \underset{\to}{f}_{i,ext} + \sum_i \sum_j \underset{\to}{f}_i^j$
    * The double sum becomes $\frac{1}{2}\sum_i \sum_j (\underset{\to}{f}_i^j + \underset{\to}{f}_i^j) = \frac{1}{2}\sum_i \sum_j (\underset{\to}{f}_i^j - \underset{\to}{f}_j^i)$ by Newton's third law
    * But since we are summing over all i and j, $\frac{1}{2}\sum_i \sum_j (\underset{\to}{f}_i^j - \underset{\to}{f}_j^i) = \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_i^j - \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_j^i = \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_i^j - \frac{1}{2}\sum_i \sum_j \underset{\to}{f}_i^j = \underset{\to}{0}$
  - Therefore $m\ddot{\underset{\to}{r}}_{\text{◕}} = \underset{\to}{f}$, which is Newton's second law on the center of mass – we've extended the law from individual particles to systems of particles
- What about the rotational equations of motion?
  - $\dot{\underset{\to}{h}}_i = m_i \dot{\underset{\to}{r}}_i \times \dot{\underset{\to}{r}}_i + m_i \underset{\to}{r}_i \times \ddot{\underset{\to}{r}}_i = m_i \underset{\to}{r}_i \times \ddot{\underset{\to}{r}}_i = \underset{\to}{r}_i \times \underset{\to}{f}_i = \underset{\to}{r}_i \times \left(\underset{\to}{f}_{i,ext} + \sum_j \underset{\to}{f}_i^j\right)$
  - So $\dot{\underset{\to}{h}} = \sum_i \dot{\underset{\to}{h}}_i = \sum_i \underset{\to}{r}_i \times \underset{\to}{f}_{i,ext} + \sum_i \sum_j \underset{\to}{r}_i \times \underset{\to}{f}_i^j = \underset{\to}{\tau} + \sum_i \sum_j \underset{\to}{r}_i \times \underset{\to}{f}_i^j$

- $\displaystyle\sum_i \sum_j \underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_i^j = \frac{1}{2}\sum_i \sum_j (\underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_i^j + \underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_i^j)$

$$= \frac{1}{2}\sum_i \sum_j (\underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_i^j - \underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_j^i)$$

$$= \frac{1}{2}\sum_i \sum_j (\underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_i^j - \underset{\rightarrow}{r}_j \times \underset{\rightarrow}{f}_i^j)$$

$$= \frac{1}{2}\sum_i \sum_j (\underset{\rightarrow}{r}_i - \underset{\rightarrow}{r}_j) \times \underset{\rightarrow}{f}_i^j$$

  * If we assume that $(\underset{\rightarrow}{r}_i - \underset{\rightarrow}{r}_j) \parallel \underset{\rightarrow}{f}_i^j$, that is, the inter-particle forces act along the lines connecting two particles, then we can make this term zero
  * This additional assumption, that equal and opposite forces act along the line connecting two particles, is the *strong form* of Newton's third law
  * This is a reasonable assumption to make because otherwise two particles in a system would keep accelerating forever
- Hence $\underset{\rightarrow}{h}^{\bullet} = \underset{\rightarrow}{\tau} = \sum_i \underset{\rightarrow}{r}_i \times \underset{\rightarrow}{f}_{i,ext}$
- Note this is not $\underset{\rightarrow}{h}_{\odot}$, but angular momentum about the inertial reference point $O_{\mathfrak{I}}$

- $\underset{\rightarrow}{h}_{\odot} = \sum_i s_i \times m_i(\underset{\rightarrow}{r}_{\odot}^{\bullet} + \underset{\rightarrow}{s}_i^{\bullet}) = \sum_i m_i \underset{\rightarrow}{s}_i \times \underset{\rightarrow}{s}_i^{\bullet} + \left(\sum_i m_i \underset{\rightarrow}{s}_i\right) \times \underset{\rightarrow}{r}_{\odot}^{\bullet} = \sum_i m_i \underset{\rightarrow}{s}_i \times \underset{\rightarrow}{s}_i^{\bullet}$
  - $\underset{\rightarrow}{h}_{\odot}^{\bullet} = \sum_i \underset{\rightarrow}{s}_i^{\bullet} \times \underset{\rightarrow}{p}_i + \sum_i \underset{\rightarrow}{s}_i \times \underset{\rightarrow}{p}_i^{\bullet} = \sum_i \underset{\rightarrow}{s}_i \times \underset{\rightarrow}{f}_{i,ext} = \underset{\rightarrow}{\tau}_{\odot}$
- Therefore $\underset{\rightarrow}{h}^{\bullet} = \underset{\rightarrow}{\tau}$ about $O_{\mathfrak{I}}$ and $\underset{\rightarrow}{h}_{\odot}^{\bullet} = \underset{\rightarrow}{\tau}_{\odot}$ about $\odot$
  - This is a special result that only holds for the center of mass!
- In general $\underset{\rightarrow}{h}_O^{\bullet} + \underset{\rightarrow}{v}_O \times \underset{\rightarrow}{p} = \underset{\rightarrow}{\tau}_O$ for a general O moving at $\underset{\rightarrow}{v}_O = \underset{\rightarrow}{r}^{OO_{\mathfrak{I}}\,\bullet}$ with respect to $O_{\mathfrak{I}}$
  - If $\underset{\rightarrow}{\rho}_i$ is the position of each particle about a moving reference point O, then each particle has inertial velocity $\underset{\rightarrow}{v} = \underset{\rightarrow}{v}_O + \underset{\rightarrow}{\rho}^{\bullet}$ so $\underset{\rightarrow}{\rho}^{\bullet} = \underset{\rightarrow}{v} - \underset{\rightarrow}{v}_O$
  - $\underset{\rightarrow}{h}^{\bullet} = \sum_i (\underset{\rightarrow}{\rho}_i \times \underset{\rightarrow}{p}_i)^{\bullet}$

$$= \sum_i m_i \underset{\rightarrow}{\rho}_i^{\bullet} \times \underset{\rightarrow}{v}_i + \sum_i m_i \underset{\rightarrow}{\rho}_i \times \underset{\rightarrow}{v}_i^{\bullet}$$

$$= \sum_i m_i(\underset{\rightarrow}{v}_i - \underset{\rightarrow}{v}_O) \times \underset{\rightarrow}{v}_i + \sum_i \underset{\rightarrow}{\rho}_i \times \underset{\rightarrow}{f}_i$$

$$= -\underset{\rightarrow}{v}_O \times \sum_i m_i v_i + \sum_i \underset{\rightarrow}{\tau}_i$$

$$= -\underset{\rightarrow}{v}_O \times \underset{\rightarrow}{p} + \underset{\rightarrow}{\tau}_O$$

  - In the inertially fixed point, $\underset{\rightarrow}{v}_O = 0$
  - At the center of mass, $\underset{\rightarrow}{p} = m\underset{\rightarrow}{r}_{\odot}^{\bullet} = m\underset{\rightarrow}{v}_O$, so when we cross it with $\underset{\rightarrow}{v}_O$, the term cancels
  - In both special frames we do not need to apply a correction
- What about work and kinetic energy?
  - $W_i = \displaystyle\int_A^B \underset{\rightarrow}{f}_i \cdot \mathrm{d}\underset{\rightarrow}{r}_i$
  - $W = \sum_i W_i = \sum_i T_i^B - \sum_i T_i^A = T_B - T_A$ by the principle of work and kinetic energy
  - $T_i = \frac{1}{2} m_i \underset{\rightarrow}{v}_i \cdot \underset{\rightarrow}{v}_i = \frac{1}{2} m_i \underset{\rightarrow}{r}_i^{\bullet} \cdot \underset{\rightarrow}{r}_i^{\bullet}$ and $\underset{\rightarrow}{r}_i^{\bullet} = \underset{\rightarrow}{r}_{\odot}^{\bullet} + \underset{\rightarrow}{s}_i^{\bullet}$, so substituting in:

- $T = \sum_i T_i$
  $= \frac{1}{2}\sum_i m_i(\dot{\vec{r}}_{\odot} \cdot \dot{\vec{r}}_{\odot} + 2\dot{\vec{s}}_i \cdot \dot{\vec{r}}_{\odot} + \dot{\vec{s}}_i \cdot \dot{\vec{s}}_i)$
  $= \frac{1}{2}mv_{\odot}^2 + \left(\sum_i m_i\dot{\vec{s}}_i\right) \cdot \vec{v}_{\odot} + \frac{1}{2}\sum_i m_i\dot{\vec{s}}_i \cdot \dot{\vec{s}}_i$
  $= \frac{1}{2}mv_{\odot}^2 + \frac{1}{2}\sum_i m_i u_i^2$
- So the total kinetic energy is the kinetic energy as if all the mass is at the center of mass, plus the kinetic energy of all the particles relative to the center of mass
- If we have a rigid body in translation, all $u_i = 0$, so the kinetic energy is just the same as if the mass is concentrated at the center of mass
  * Note this does not apply in rotation – in that case the second term would result in rotational kinetic energy

Summary

For a grammar of particles, the total momentum and angular momentum are defined as:

$$\vec{p} = \sum_i \vec{p}_i = \sum_i m_i\dot{\vec{r}}_i \qquad \vec{h} = \sum_i \vec{h}_i = \sum_i m_i\vec{r}_i \times \dot{\vec{r}}_i$$

The centre of mass, located at $\vec{r}_{\odot} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$, satisfies:

$$m\vec{r}_{\odot} = \vec{p} \qquad m\ddot{\vec{r}}_{\odot} = \vec{f} = \sum_i \vec{f}_{i,ext}$$

The total angular momentum about O, moving at $\vec{v}_O$ relative to inertial space, in general satisfies:

$$\dot{\vec{h}}_O + \vec{v}_O \times \vec{p} = \vec{\tau}_O = \sum_i \vec{\rho}_i \times \vec{f}_{i,ext}$$

When $O = \odot$ or inertially fixed, this reduces to $\dot{\vec{h}} = \vec{\tau}$.

D'Alembert's Principle

- Let $\vec{\rho}_i$ be the position of particle i with respect to O, an arbitrary, possibly accelerating reference point, so $\vec{r}_i = \vec{r}^{OO_{\mathfrak{I}}} + \vec{\rho}_i$
- $m\ddot{\vec{\rho}}_{\odot} = \vec{f} \implies m(\ddot{\vec{r}}^{OO_{\mathfrak{I}}} + \ddot{\vec{\rho}}_{\odot}) = \vec{f}$
- Let $\ddot{\vec{r}}^{OO_{\mathfrak{I}}} = \vec{a}_O$ be the acceleration of O with respect to $O_{\mathfrak{I}}$; then $m\ddot{\vec{\rho}}_{\odot} = \vec{f} - m\vec{a}_O$
  - Note we've made Newton's second law work by applying a "reversed inertial force"
  - This is the essence of d'Alembert's Principle – the ability to transform ourselves into a noninertial frame but still use Newton's second law

Lecture 10, Oct 10, 2023

D'Alembert's Principle

- Consider an inertially fixed point $O_{\mathfrak{I}}$, an inertially moving point O and a grammar of particles $\mathcal{P}_i$, with positions $\vec{r}_i$ relative to $O_{\mathfrak{I}}$ and $\vec{\rho}_i$ relative to O
- How can we accommodate Newton's second law in a noninertial frame?

- $\vec{r}_i = \vec{r}^{O\mathfrak{I}} + \vec{\rho}_i$
- The equations of motion are $m_i \ddot{\vec{r}}_i = \vec{f}_{i,ext} + \sum_j \vec{f}_i^j = m_i(\ddot{\vec{\rho}}_i + \ddot{\vec{r}}^{O\mathfrak{I}})$
  - Let $\vec{a}_O = \ddot{\vec{r}}^{O\mathfrak{I}}$ which is the acceleration of O with respect to $O_\mathfrak{I}$
  - $m_i \ddot{\vec{\rho}}_i = \vec{f}_{i,ext} + \sum_i \vec{f}_i^j - m_i \vec{a}_O$
  - Therefore $m\ddot{\vec{\rho}}_\otimes = \vec{f} - m\vec{a}_O$
  - Let $\vec{\pi}_i = m_i \dot{\vec{\rho}}_i$ be the momentum as seen in O
  - $\vec{\pi} = m\dot{\vec{\rho}}_\otimes$
  - Therefore $\dot{\vec{\pi}} = \vec{f} - m\vec{a}_O$
  - Notice that the rate of change momentum as observed in O is the total force, plus a reversed inertial force
- For the angular momentum: $\vec{\eta}_i = \vec{\rho}_i \times \vec{\pi}_i$
  - The total momentum is then $\vec{\eta}_0 = \sum_i m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$
  - $\dot{\vec{\eta}}_0 = \sum_i \vec{\rho}_i \times (m_i \ddot{\vec{\rho}}_i) = \sum_i \vec{\rho}_i \times \left(\vec{f}_{i,ext} + \sum_j \vec{f}_i^j - m_i \vec{a}_O\right)$
  - $\dot{\vec{\eta}}_0 = \vec{\tau}_O - \left(\sum_i m_i \vec{\rho}_i\right) \times \vec{a}_O = \vec{\tau}_O - m\vec{\rho}_\otimes \times \vec{a}_O = \vec{\tau}_O + \vec{\rho}_\otimes \times (-m\vec{a}_O)$

Definition

D'Alembert's principle: The classical laws of mechanics can be applied in a linearly accelerating frame if reversed inertial forces are applied to the center of mass of a grammar of particles.

The Rocket Problem

- Consider a rocket with mass m moving at velocity $\vec{v}$; if $\vec{f} = \dot{\vec{p}}$, then do we have $\vec{f} = \dot{m}\vec{v} + m\dot{\vec{v}}$, since the mass is changing for the rocket?
  - The problem is that we failed to consider the part of the mass that was ejected
- At time t, the rocket has mass m and velocity $\vec{v}$; at time $t + \mathrm{d}t$, the rocket has mass $m + \mathrm{d}m$ and velocity $\vec{v} + \mathrm{d}\vec{v}$, where $-\mathrm{d}m$ is the amount of mass that was ejected at a velocity $\vec{v} + \vec{v}_{ex}$
- $\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \implies \mathrm{d}\vec{p} = \vec{f}\,\mathrm{d}t = \vec{p}(t + \mathrm{d}t) - \vec{p}(t)$
  - $\vec{p}(t) = m\vec{v}$
  - $\vec{p}(t + \mathrm{d}t) = (m + \mathrm{d}m)(\vec{v} + \mathrm{d}\vec{v}) + (-\mathrm{d}m)(\vec{v} + \vec{v}_{ex})$
  - Under a first order approximation, this reduces to $\mathrm{d}\vec{p} = m\mathrm{d}\vec{v} - \vec{v}_{ex}\mathrm{d}m = \vec{f}\,\mathrm{d}t$
  - Therefore $m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \vec{v}_{ex}\frac{\mathrm{d}m}{\mathrm{d}t} = \vec{f} \implies m\dot{\vec{v}} = \vec{f} + \dot{m}\vec{v}_{ex}$
- Comparing this result to what we have before, the difference is that instead of $-\vec{v}_{ex}$, in our first incorrect result we had $\vec{v}$
- Example: consider an open-top railway car with mass m_0 and velocity v_0, which is collecting rain at a rate $\dot{m}$; what is the velocity of the car after a given time, where an amount of rain m has fallen?
  - Suppose there is an amount of water m in the car at time t, then $p(t) = (m_0 + m)v$ and $p(t + \mathrm{d}t) = (m_0 + m + \mathrm{d}m)(v + \mathrm{d}v)$
  - Since the rain is falling vertically, $p(t) = p(t + \mathrm{d}t) \implies (m_0 + m)v = (m_0 + m + \mathrm{d}m)(v + \mathrm{d}v)$
  - $v\,\mathrm{d}m + (m_0 + m)\mathrm{d}v = 0 \implies \frac{\mathrm{d}v}{v} = -\frac{\mathrm{d}m}{m_0 + m}$ with initial conditions $m = 0, v = v_0$ at $t = 0$
  - Integrating both sides: $\ln\frac{v}{v_0} = -\ln\frac{m_0 + m}{m_0} \implies v = v_0\frac{m}{m_0 + m} \implies (m_0 + m)v = m_0 v_0$
- Example: if the car is dripping so that the water level stays the same, what is the velocity now?
  - $p(t) = m_0 v$

  - $p(t+\mathrm{d}t) = m_0(v+\mathrm{d}v) + (v+\mathrm{d}v)\mathrm{d}m$ where $-\mathrm{d}m$ is the amount of water falling out
  - Again $p(t+\mathrm{d}t) - p(t) = 0 \implies m_0\,\mathrm{d}v + v\,\mathrm{d}m = 0 \implies \frac{\mathrm{d}v}{v} = -\frac{\mathrm{d}m}{m_0} \implies \ln\frac{v}{v_0} = -\frac{m}{m_0} \implies v = v_0 e^{-\frac{m}{m_0}}$

Lecture 11, Oct 12, 2023

Orbital Motion and Gravity

- Kepler's laws:
  1. The orbits of planets are ellipses with the sun at one focus
  2. An orbiting planet sweeps out an equal area in equal time, i.e. $\frac{\mathrm{d}A}{\mathrm{d}t} = c$
  3. The square of the orbital period of a planet is proportional to the cube of the mean distance from the sun, i.e. $T^2 \propto a^3$
- Kepler tells us the shape of the orbits without need for epicycles, but still does not tell us the fundamental force that causes such motion
- Newton's law of gravitation: $\underset{\to}{f}{}_b^a = \frac{Gm_am_b}{r_3}\underset{\to}{r}{}_b^a$ where $G = 6.67\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is the universal gravitation constant
  1. The force of gravity is central
  2. The force of gravity varies according to the inverse-square law
  3. The force of gravity is universal
- Kepler's second law can be derived directly from the fact that gravity is a central force
- Kepler's third law implies that this central force varies according to the inverse square law
- The universality of gravity can be determined by comparing the acceleration of an object on earth to the acceleration of the moon, after taking into account the inverse square law
- Inside a hollow shell, there is no gravitational force due to the inverse square law
- Inside a solid sphere, the gravitational force varies proportional to r (since the mass varies as r^3 and the gravitational force varies as r^{-2})
  - This means if we dug a hole through the earth and dropped a particle, it would exhibit simple harmonic motion, since the force is proportional to distance from the core

Lecture 12, Oct 17, 2023

Recovering Kepler's Laws

- Using Newton's law of gravitation, we will attempt to recover Kepler's 3 laws
- Consider a grammar of 2 particles a and b; the particles have positions $\underset{\to}{r}_a$ and $\underset{\to}{r}_b$ relative to the center of mass; let $\underset{\to}{r} = \underset{\to}{r}_a - \underset{\to}{r}_b$ be the vector connecting the two particles
  - Since we have no external forces acting on the system, $m\ddot{\underset{\to}{r}}_{◕} = \underset{\to}{f}_{ext} = \underset{\to}{0}$, so ◕ is not accelerating with respect to inertial frame, i.e. the ◕ frame is an inertial frame
- $\underset{\to}{f}{}_a^b = -\frac{Gm_am_b}{r^3}\underset{\to}{r}$ but also $\underset{\to}{f}{}_a^b = m_a\ddot{\underset{\to}{r}}_a, \underset{\to}{f}{}_b^a = m_b\ddot{\underset{\to}{r}}_b = -\underset{\to}{f}{}_a^b$
  - $\ddot{\underset{\to}{r}} = \ddot{\underset{\to}{r}}_a - \ddot{\underset{\to}{r}}_b = \frac{1}{m_a}\underset{\to}{f}{}_a^b - \frac{1}{m_b}\underset{\to}{f}{}_b^a$
  - Using this we get $\ddot{\underset{\to}{r}} = -\frac{\mu}{r^3}\underset{\to}{r}$ where $\mu = G(m_a + m_b)$
- Therefore the relative motion of the bodies is given by $\ddot{\underset{\to}{r}} = -\frac{\mu}{r^3}\underset{\to}{r}$
  - This means instead of the motion of two bodies, we can fix b and consider the relative motion of a, with its mass replaced by the reduced mass $m = \frac{m_am_b}{m_a + m_b}$
- Consider the total angular momentum (about the centre of mass) $\underset{\to}{h} = m_a\underset{\to}{r}_a \times \dot{\underset{\to}{r}}_a + m_b\underset{\to}{r}_b \times \dot{\underset{\to}{r}}_b$
  - $\dot{\underset{\to}{h}} = m_a\underset{\to}{r}_a \times \ddot{\underset{\to}{r}}_a + m_b\underset{\to}{r}_b \times \ddot{\underset{\to}{r}}_b$ (product rule terms cancel)
  - $\dot{\underset{\to}{h}} = \underset{\to}{r}_a \times \underset{\to}{f}{}_a^b + \underset{\to}{r}_b \times \underset{\to}{f}{}_b^a = \underset{\to}{r}_a \times (k\underset{\to}{r}) + \underset{\to}{r}_b \times (k\underset{\to}{r}) = 0$

- Note this is just conservation of angular momentum

- $\underrightarrow{r} \cdot \underrightarrow{h} = m_a \underrightarrow{r} \cdot (\underrightarrow{r}_a \times \underrightarrow{r}_a^{\cdot}) + m_b \underrightarrow{r} \times (\underrightarrow{r}_b \times \underrightarrow{r}_b^{\cdot}) = 0$
  - Note that $\underrightarrow{r}_a \times \underrightarrow{r}_a^{\cdot}$ is normal to $\underrightarrow{r}_a$, which is parallel to $\underrightarrow{r}$; therefore the dot product with $\underrightarrow{r}$ is zero for both terms
  - Therefore $\underrightarrow{r}$ is always normal to $\underrightarrow{h}$, but $\underrightarrow{h}$ is constant, so $\underrightarrow{r}$ is always in some fixed plane, so now we can reduce the problem down to 2 dimensions
- Since the motion is 2-dimensional, we will use polar coordinates (r, θ) to express $\underrightarrow{r}$
- Let the (noninertial) orbital frame $\underrightarrow{\mathcal{F}}_o$ such that $\underrightarrow{o}_1$ is in the same direction as the vector connecting the two masses and $\underrightarrow{o}_2$ is in the plane of motion of $\underrightarrow{r}$
  - $\underrightarrow{r} = \underrightarrow{\mathcal{F}}_o^T \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}, \underrightarrow{\omega} = \underrightarrow{\mathcal{F}}_o^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$
  - $\underrightarrow{r}^{\cdot\cdot} = \underrightarrow{r}^{\circ\circ} + 2\underrightarrow{\omega} \times \underrightarrow{r}^{\circ} + \underrightarrow{\omega}^{\circ} \times \underrightarrow{r} + \underrightarrow{\omega} \times (\underrightarrow{\omega} \times \underrightarrow{r}) = -\frac{\mu}{r^3}\underrightarrow{r}$
  - We can now expand this out in the orbital frame and obtain the equations of motion
- $\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\dfrac{\mu}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$ are the 2-body orbital equations of motion
  - Note the multiplying the second equation by r and integrating gives $\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$, which corresponds to conservation of angular momentum (ignoring mass) ($r\dot{\theta}$ being the tangential velocity, with moment arm r)
  - This can be solved by making the substitution $r = \frac{1}{u}$, which we leave as an exercise to the reader
- We can also solve the equation of motion using vectors:
  - Starting with the equation of motion, we can cross both sides with $\underrightarrow{h}$
  - $\underrightarrow{r}^{\cdot\cdot} \times \underrightarrow{h} = -\frac{\mu}{r^3}\underrightarrow{r} \times \underrightarrow{h}$
  - $\frac{\mathrm{d}}{\mathrm{d}t}(\underrightarrow{r}^{\cdot} \times \underrightarrow{h}) = -\frac{\mu}{r^3}\underrightarrow{r} \times (\underrightarrow{r} \times \underrightarrow{r}^{\cdot}) = -\frac{\mu}{r^3}((\underrightarrow{r} \cdot \underrightarrow{r}^{\cdot})\underrightarrow{r} - (\underrightarrow{r} \cdot \underrightarrow{r})\underrightarrow{r}^{\cdot}) = -\frac{\mu}{r_3}(r\dot{r}\underrightarrow{r} - \underrightarrow{r}^2\underrightarrow{r}^{\cdot}) = -\mu\left(\frac{\dot{r}}{r^2}\underrightarrow{r} - \frac{1}{r}\underrightarrow{r}^{\cdot}\right) = \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\underrightarrow{r}}{r}\right)$
    - Note $\underrightarrow{u} \times (\underrightarrow{v} \times \underrightarrow{w}) = (\underrightarrow{v} \cdot \underrightarrow{w})\underrightarrow{v} - (\underrightarrow{u} \cdot \underrightarrow{v})\underrightarrow{w}$
    - $r\dot{r} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}r^2 + = \frac{2}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\underrightarrow{r} \cdot \underrightarrow{r}) = \frac{1}{2}(\underrightarrow{r}^{\cdot} \cdot \underrightarrow{r} + \underrightarrow{r} \cdot \underrightarrow{r}^{\cdot}) = \underrightarrow{r} \cdot \underrightarrow{r}^{\cdot}$
  - $\underrightarrow{r}^{\cdot} \times \underrightarrow{h} = \mu\left(\frac{\underrightarrow{r}}{r} + \underrightarrow{\varepsilon}\right)$
  - $\underrightarrow{r} \cdot \underrightarrow{r}^{\cdot} \times \underrightarrow{h} = \mu(r + \underrightarrow{r} \cdot \underrightarrow{\varepsilon}) = \mu(r + \|\underrightarrow{r}\|\|\underrightarrow{\varepsilon}\|\cos\theta) = \mu r(1 + \varepsilon\cos\theta)$
  - $\underrightarrow{h} \times (\underrightarrow{r} \times \underrightarrow{r}^{\cdot}) = \underrightarrow{h} \cdot \underrightarrow{h} = h^2$
  - $h^2 = \mu r(1 + \varepsilon\cos\theta)$ or $r = \dfrac{\frac{h^2}{\mu}}{1 + \varepsilon\cos\theta}$
- $r = \dfrac{\frac{h^2}{\mu}}{1 + \varepsilon\cos\theta}$ is the shape of the orbit, which describes a general conic section (ellipse, parabola, hyperbola)
  - ε is the *eccentricity* of the conic section, $\varepsilon = 0$ gives a circle, $0 < \varepsilon < 1$ gives an ellipse, $\varepsilon = 1$ gives a parabola, $1 < \varepsilon$ gives a hyperbola
  - This gives us Kepler's first law
  - But also, we know that orbits can also be parabolas or hyperbolas
  - If we took the energy $e = \frac{1}{2}v^2 - \frac{\mu}{r}$ we can express everything in terms of it
- If between time t and $t + \mathrm{d}t$ the mass travelled an angle $\mathrm{d}\theta$, then the area swept out is $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$
  - $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h$, but we know this is constant from earlier
  - This gives us Kepler's second law

- $t = \sqrt{\dfrac{a^3}{\mu}}(E - \varepsilon \sin E)$
    - The period is $T = 2\pi\sqrt{\dfrac{a^3}{\mu}}$
    - This is Kepler's third law

Lecture 13, Oct 19, 2023

Perturbation Theory

- Perturbation theory deals with the changes in a function (e.g. an orbit) resulting from a small change
- Let $x(t) = x_0(t) + \Delta x(t)$, where $x(t)$ is some nominal solution and $\Delta x(t)$ is some small disturbance; our goal is to get the response $f(x)$ in terms of Δx only, since we might not have a closed-form expression for f
- Recall the equations of motion: $\begin{cases} f_1(r, \dot{r}, \ddot{r}, \omega, \dot{\omega}) = \ddot{r} - r\omega^2 = -\dfrac{\mu}{r^2} \\ f_2(r, \dot{r}, \ddot{r}, \omega, \dot{\omega}) = r\dot{\omega} + 2\dot{r}\omega = 0 \end{cases}$
- Take some some nominal/reference solutions $r_0(t), \omega_0(t)$ so that $\begin{cases} r(t) = r_0(t) + \Delta r(t) \\ \omega(t) = \omega_0(t) + \Delta\omega(t) \end{cases}$
- Consider a satellite in orbit; we fire thrusters such that at time $t = 0$, we have an instantaneous increase in velocity Δv tangential to the orbit
    - The resulting orbit will be slightly elliptical
- Plug in reference solution to first equation: $(\ddot{r}_0 + \Delta\ddot{r}) - (r_0\Delta r)(\omega_0 + \Delta\omega)^2 = -\dfrac{\mu}{(r_0 + \Delta r)^2}$
    - Expand and ignore all terms second order and above
    - $(\ddot{r}_0 + \Delta\ddot{r}) - r_0\omega_0^2 + 2r_0\omega_0\Delta\omega + \omega_0^2\Delta r = -\dfrac{\mu}{r_0^2\left(1 + \frac{\Delta r}{r_0}\right)} = -\dfrac{\mu}{r_0^2}\left(1 - 2\dfrac{\Delta r}{r_0}\right)$
    - $\ddot{r}_0 - r_0\omega_0^2 + \Delta\ddot{r} - 2r_0\omega_0\Delta\omega - \omega_0^2\Delta r = -\dfrac{\mu}{r_0^2} + 2\dfrac{\mu}{r_0^3}\Delta r$
    - Compare this with the equation of motion, we can subtract $\ddot{r}_0 - r_0\omega_0^2 = -\dfrac{\mu}{r_0^2}$ from both sides
    - $\Delta\ddot{r} - 2r_0\omega_0\Delta\omega - 3\omega_0^2\Delta r = 0$
- The second equation gives $r_0\Delta\dot{\omega} + 2\omega_0\Delta\dot{r} = 0$, with initial conditions $\Delta r(0) = 0, \Delta\dot{r}(0) = 0, \Delta\omega(0) = \dfrac{\Delta v}{r_0}$
    - Integrate directly: $r_0\Delta\omega + 2\omega_0\Delta r = C = \Delta v$
    - The first equation becomes: $\Delta\ddot{r} + 4\omega_0^2\Delta r - 3\omega_0^2\Delta r = 2\omega_0\Delta v$
    - $\Delta\ddot{r} + \omega_0^2\Delta r = 2\omega_0\Delta v$
    - Particular solution: $\Delta r(t) = \dfrac{2\Delta v}{\omega_0}$
    - Homogeneous solution: $c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$
    - Solve for initial conditions: $\Delta r(t) = \dfrac{2\Delta v}{\omega_0}(1 - \cos(\omega_0 t))$
- The resulting orbit is slightly elliptical, and at $t = 0$ we're at the point on the ellipse closest to the focus

Gravity Boosting/Braking

- Consider a body with velocity $\underset{\rightarrow}{v}_p$; a spacecraft of velocity $\underset{\rightarrow}{v}_s^-$ approaches it from far away, and we want to know the velocity of the spacecraft $\underset{\rightarrow}{v}_s^+$ after escaping the planet's gravity
- Shift into the planet's reference frame; the velocity of the spacecraft in this frame is $\underset{\rightarrow}{u}_s^- = -\underset{\rightarrow}{v}_p + \underset{\rightarrow}{v}_s^-$
- Since the spacecraft is coming from infinity, the orbital shape is effectively hyperbolic
    - The spacecraft will be entering and leaving with the same magnitude of velocity in the planet's frame, $\|\underset{\rightarrow}{u}_s^-\| = \|\underset{\rightarrow}{u}_s^+\|$
- If we now shift into the sun's reference frame, the spacecraft leaves with velocity $\underset{\rightarrow}{v}_s^+ = \underset{\rightarrow}{u}_s^+ + \underset{\rightarrow}{v}_p$

  - Depending on the direction that the vectors are arranged, $\vec{v}_s^+$ can be much faster or slower than $\vec{v}_s^-$
  - If the spacecraft passes behind the planet, it will speed up; if it passes in front of the planet, it will slow down
- This is called gravity boosting or braking (aka the slingshot effect)

Example Problem

- Given that the orbital shape of a body is $r = a\sqrt{\cos 2\theta}$, show that in order to create this orbital shape, the force satisfies $f \propto \frac{1}{r^7}$
- For a general central force we have $\begin{cases} \ddot{r} - r\dot{\theta}^2 = \frac{f}{m} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$
- Now we can simply substitute in r: $\begin{cases} \dot{r} = -\frac{a\dot{\theta}\sin 2\theta}{\sqrt{\cos 2\theta}} \\ \ddot{r} = -\frac{a\ddot{\theta}\sin 2\theta}{\sqrt{\cos 2\theta}} - \frac{a\dot{\theta}\sin 2\theta}{\cos^{\frac{3}{2}} 2\theta} - 2a\dot{\theta}^2\sqrt{\cos 2\theta} \end{cases}$
  - Using the second equation, $\dot{\theta} = \frac{h}{r^2}$ and $\ddot{\theta} = -\frac{2h}{r^3}\dot{r} = \frac{2\dot{\theta}^2 \sin 2\theta}{\cos 2\theta}$
  - This allows us to express everything in terms of r in the first equation

Lecture 14, Oct 24, 2023

Virtual Work and D'Alembert's Principle

- Consider a grammar of particles in static equilibrium; we will have that $\vec{f}_i = \vec{0}$; then for any small displacement $\delta\vec{r}_i = 0$, so $\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$
  - We define this quantity as the *virtual work* $\widehat{\delta W}$ (note the ligature since work is not path-independent)
  - We call $\delta\vec{r}_i$ a *virtual displacement*
- In general we can write $\vec{f}_i = \vec{f}_{i,app} + \vec{f}_{i,\square}$, where $\vec{f}_{i,app}$ is an applied force and $\vec{f}_{i,\square}$ is a constraint force
  - Then $\widehat{\delta W} = \sum_i \vec{f}_{i,app} \cdot \delta\vec{r}_i + \sum_i \vec{f}_{i,\square} \cdot \delta\vec{r}_i = \sum_i \vec{f}_{i,app} \cdot \delta\vec{r}_i$ because constraint forces do no work, provided that the $\delta\vec{r}_i$ are consistent with the geometry of the system
    - This is the *principle of virtual work* and is an assumption that we make
- What about particles in dynamic equilibrium?
  - $\vec{f}_i = m\ddot{\vec{r}}$
  - So we just need to consider $\vec{f}_i - m_i\ddot{\vec{r}}_i$ as the total force – according to d'Alembert's principle
  - $\widehat{\delta W} = \sum_i (\vec{f}_{i,app} - m_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0$ by the same reasoning and assumptions above, as long as $\delta\vec{r}_i$ is consistent with the system's constraints
    - This is known as *d'Alembert's principle* (again)
  - Note that since we no longer have to consider constraint forces we will drop the subscript

Lagrangian Mechanics

- Consider an independent set of coordinates $q_1, q_2, \dots, q_n$ where n is the number of degrees of freedom
  - The coordinates must be complete, i.e. satisfy $\vec{r}_i = \vec{r}_i(q_{11}, q_{12}, \dots, q_n, t)$; the position of any particle must be expressible in terms of the generalized coordinates
  - These coordinates are called *generalized coordinates*, because the do not have to be Cartesian; instead they can be displacements or angles etc
  - We aim to obtain equations of motions in these generalized coordinates only

- Any permissible virtual displacement can then be given in terms of these coordinates: $\delta \underset{\rightarrow}{r_i} = \sum_k \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} \delta q_k$
  - in this form it is clear that the virtual displacements are permissible
    - Note that even though $\underset{\rightarrow}{r_i}$ can be dependent on time, the virtual displacements $\delta \underset{\rightarrow}{r_i}$ are "frozen" in time; this is why we don't need to consider $\frac{\partial \underset{\rightarrow}{r_i}}{\partial t}$
- $\sum_i \underset{\rightarrow}{f_i} \cdot \delta \underset{\rightarrow}{r_i} - \sum_i m_i \ddot{\underset{\rightarrow}{r_i}} \cdot \delta \underset{\rightarrow}{r_i} = 0$
  - $\sum \underset{\rightarrow}{f_i} \cdot \delta \underset{\rightarrow}{r_i} = \sum_i \sum_k \underset{\rightarrow}{f_i} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} \delta q_k = \sum_k Q_k \delta q_k$ where $Q_k = \sum_i \underset{\rightarrow}{f_i} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}$
    - Q_k are referred to as the *generalized forces*
  - $\sum_i m_i \ddot{\underset{\rightarrow}{r_i}} \cdot \delta \underset{\rightarrow}{r_i} = \sum_i \sum_j m_i \ddot{\underset{\rightarrow}{r_i}} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} \delta q_k = \sum_i \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \dot{\underset{\rightarrow}{r_i}} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} \right) - m_i \dot{\underset{\rightarrow}{r_i}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \underset{\rightarrow}{r_i}}{\partial q_i} \right) \right] \delta q_k$
  - Note $\frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} = \frac{\partial \underset{\rightarrow}{v_i}}{\partial \dot{q}_k}$ and $\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} \right) = \frac{\partial \underset{\rightarrow}{v_i}}{\partial q_k}$ so the stuff in brackets becomes $\frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \underset{\rightarrow}{v_i} \cdot \frac{\partial \underset{\rightarrow}{v_i}}{\partial \dot{q}_i} \right) - m_i \underset{\rightarrow}{v_i} \cdot \frac{\partial \underset{\rightarrow}{v_i}}{\partial q_k} = \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_k} \left(\frac{1}{2} m_i \underset{\rightarrow}{v_i} \cdot \underset{\rightarrow}{v_i} \right) = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial \dot{q}_k} T_i \right) - \frac{\partial}{\partial q_k} T_i$
  - Together we have $\sum_k Q_k \delta q_k - \sum_k \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial}{\partial \dot{q}_k} \left(\sum_i T_i \right) \right) - \frac{\partial}{\partial q_k} \left(\sum_i T_i \right) \right] \delta q_k = 0$
  - $\sum_k \left[Q_k - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_k} \right) + \frac{\partial T}{\partial q_k} \right] \delta q_k = 0$
  - Since δq_k are all independent, we can choose each one independently and arbitrarily; therefore we need $Q_k \delta q_k - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_k} \right) + \frac{\partial T}{\partial q_k} = 0$ for all k
- $\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_k} \right) + \frac{\partial T}{\partial q_k} = Q_k$
  - $Q_k = \sum_i \underset{\rightarrow}{f_i} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}$
  - Split the forces into conservative and non-conservative parts $\underset{\rightarrow}{f_i} = -\vec{\nabla} V_i + \underset{\rightarrow}{f_{i,\triangle}}$
  - Then $Q_k = -\sum_i \vec{\nabla} V_i \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} + \sum_i f_{i,\triangle} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k} = -\frac{\partial V}{\partial q_k} + Q_{k,\triangle}$
    - $Q_{k,\triangle}$ are the generalized non-conservative forces
  - Plugging this back in we get $\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_k} \right) + \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} + Q_{k,\triangle}$
    - Assuming V is a function of position only, $\frac{\partial V}{\partial \dot{q}_k} = 0$, so we have $\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial}{\partial \dot{q}_k} (T - V) \right) + \frac{\partial}{\partial q_k} (T - V) = Q_{k,\triangle}$
  - Letting $L = T - V$, we get $\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle}$
- $\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle}$ is *Lagrange's equation* (aka the Euler-Lagrange equation)
  - $Q_{k,\triangle} = \sum_i \underset{\rightarrow}{f_{i,\triangle}} \cdot \frac{\partial \underset{\rightarrow}{r_i}}{\partial q_k}$ is the generalized non-conservative force
  - $L = T - V$ is the *Lagrangian*, the difference between the kinetic and potential energies (summed across all particles)
    - Note that when we take the partials with respect to $\dot{q}_k$ and q_k, we treat these two as independent
  - For a system with n degrees of freedom, there are n Lagrange's equations
  - If we have no non-conservative forces, then the equation equals zero
  - Lagrange's equation is equivalent to Newton's laws – we can replace $\underset{\rightarrow}{f} = m\ddot{\underset{\rightarrow}{r}}$ by this formulation to yield identical results
- Note that since we started with Newton's second law, this can only be applied in an inertial frame

- There is a way around this by considering the potential to be velocity-dependent

- Example: pendulum with length l, mass m and angle θ
  - $q_1 = \theta$
  - $T = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$
  - $V = -mg\cos\theta$ (gravity, with the pivot as reference height)
    - * Note it doesn't matter what we take as the reference here because we always take the partial derivative of the Lagrangian, so constant factors in energy disappear as expected
  - Assume that there are no non-conservative forces
  - $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$
  - $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$
  - Plugging in, we get $ml^2\ddot{\theta} - (-mgl\sin\theta) = 0 \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0$, exactly what we get with Newtonian mechanics

Lecture 15, Oct 26, 2023

Example: Spherical Pendulum

- Consider a pendulum of mass m, length l at an angle θ from the vertical and ϕ from the x axis; what are the equations of motion
- Our generalized coordinates are $q_1 = \phi, q_2 = \theta$
- To get the kinetic energy we break up the velocity into two components
  - $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((l\dot{\theta})^2 + (l\dot{\phi}\sin\theta)^2\right) = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$
- Using the pivot as zero, the potential energy due to gravity is $V = mgl\cos\theta$
- No $Q_{k,\triangle}$ because there are no non-conservative forces in our problem
- $L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta$
- Compute the derivatives:
  - $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) ml^2\ddot{\theta}$
  - $\frac{\partial L}{\partial \theta} = ml^2\dot{\phi}^2\sin\theta\cos\theta + mgl\sin\theta$
  - $\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta$
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ml^2\ddot{\phi}\sin^2\theta - 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta$
  - $\frac{\partial L}{\partial \phi} = 0$
- The two equations are:
  - $ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0$
  - $ml^2\ddot{\phi}\sin^2\theta - 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0$
- Note because $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$, we can immediately say $\frac{\partial L}{\partial \dot{\phi}}$ is a constant
  - In this case we refer to ϕ as an *ignorable*, or *cyclical coordinate*
  - Physically, $ml^2\dot{\phi}\sin^2\theta$ is the angular momentum about the vertical axis, which is conserved, hence it is a constant
  - $\frac{\partial L}{\partial \dot{q}_k}$ is a *generalized momentum*

Constraints – Method of Lagrange Multipliers

- Consider a hoop of radius a rolling without slipping at an angle ϕ down a ramp, at a distance x
  - In this case we only have 1 independent coordinate – we can't move x and ϕ independently of each other
  - $\mathrm{d}x = a\mathrm{d}\phi \implies x = a\phi + x_0$
  - Starting from the differential form we were able to integrate this to get an expression relating x and ϕ; this is not always possible
  - If we had an expression involving x, we can simply replace it by an expression of ϕ
- Now consider that hoop rolling on a 2D surface, at a direction of θ with respect to the x axis
  - $\mathrm{d}x = a\mathrm{d}\phi\cos\theta, \mathrm{d}y = a\mathrm{d}\phi\sin\theta$
  - We have 2 independent coordinates since the change in θ and ϕ dictate the change in x and y
  - But we can no longer directly integrate our differential constraints since x and y depend on the entire history of θ and ϕ
  - Even though we only have 2 independent coordinates, we cannot write x and y independently of ϕ and θ, so we can't substitute x and y for ϕ and θ anymore
- In general, integrable constraints can be written as $\varphi(x, \phi) = 0$ and are referred to as *holonomic* constraints; if they can't be integrated, they are *non-holonomic*
  - Note that inequality constraints are holonomic
  - Holonomic constraints tell you "where" you can go – they limit the space of coordinates to a subspace that we can reach
  - Non-holonomic constraints tell you "how" you can go – they limit the possible paths we can take through the coordinate space
- Recall that from $\sum_k \left[Q_k - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k}\right]\delta q_k = 0$, we concluded that the part inside the brackets was zero for all k because the δq_k are independent; but if we have dependent coordinates, we can no longer do this
- Consider a series of (linearly independent) constraints in the form $\sum_k \Xi_{jk}\delta q_k = 0$ for $j = 1, \dots, m$
  - We can use the method of *Lagrange multipliers*
  - Multiply each constraint by λ_j, so $\sum_j \lambda_j \sum_k \Xi_{jk}\delta q_k = 0$
  - $\sum_{k=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_{k,\triangle} - \sum_{j=1}^{m}\lambda_j\Xi_{jk}\right]\delta q_k = 0$
  - Without loss of generality, let $q_k, k = 1, \dots, m$ be dependent on $q_k, k = m+1, \dots, n$, which are independent
  - We can choose λ_j such that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle} + \sum_{j=1}^{m}\lambda_j\Xi_{jk}$ for $k = 1, \dots, m$
    * Now $\sum_{k=m+1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_{k,\triangle} - \sum_{j=1}^{m}\lambda_j\Xi_{jk}\right]\delta q_k = 0$
    * But we said that q_k for $k = m+1, \dots, n$ are independent, so we can apply the same argument as before
- $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k,\triangle} + \sum_{j=1}^{m}\lambda_j\Xi_{jk}$ applies for all coordinates, regardless of independence
  - Note that we have $n+m$ unknowns (the q_k and λ_j), which are matched by our $n+m$ equations (n Lagrange equations and m constraints)
  - In general the constraints are $\sum_{k=1}^{n}\Xi_{jk}\mathrm{d}q_k + \Xi_{jt}\mathrm{d}t = 0$
    * Dividing by $\mathrm{d}t$, $\sum_{k=1}^{n}\Xi_{jk}\dot{q}_k + \Xi_{jt} = 0$, which are our m constraint equations

    - This is known as the *Pfaffian form* of the constraints

Example: Atwood's Machine

- Consider two masses m_1, m_2 hung over a pulley with mass m_p, which is concentrated at the circumference (so we don't need to worry about moment of inertia)
- The height of the masses are z_1, z_2; the pulley is rotating by an angle θ, and let $z_3 = a\theta$ where a is the radius of the pulley
- We have 1 degree of freedom, but we will use all 3 coordinates and use 2 constraints:
  - $z_1 = a\theta = z_3 = -z_2 \implies \begin{cases} \mathrm{d}z_1 - \mathrm{d}z_3 = 0 \\ \mathrm{d}z_2 + \mathrm{d}z_3 = 0 \end{cases}$
  - Therefore $\Xi_{11} = 1, \Xi_{12} = 0, \Xi_{13} = -1; \Xi_{21} = 0, \Xi_{22} = 1, \Xi_{23} = 1$
- Kinetic and potential energy:
  - $T = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 + \frac{1}{2}m_p\dot{z}_3^2$
  - $V = m_1gz_1 + m_2gz_2$
- Derivatives:
  - $\frac{\partial L}{\partial \dot{z}_1} = m_1\dot{z}_1, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_1}\right) = m_1\ddot{z}_1, \frac{\partial L}{\partial z_1} = -m_1g$
  - $\frac{\partial L}{\partial \dot{z}_2} = m_2\dot{z}_2, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_2}\right) = m_2\ddot{z}_2, \frac{\partial L}{\partial z_2} = -m_2g$
  - $\frac{\partial L}{\partial \dot{z}_3} = m_p\dot{z}_3, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_3}\right) = m_p\ddot{z}_3, \frac{\partial L}{\partial z_3} = 0$
  - Assume no non-conservative forces
- Lagrange's equations:
  - $m_1\ddot{z}_1 + m_1g = \lambda_1\Xi_{11} + \lambda_2\Xi_{21} = \lambda_1$
  - $m_2\ddot{z}_2 + m_2g = \lambda_1\Xi_{12} + \lambda_2\Xi_{22} = \lambda_2$
  - $m_p\ddot{z}_3 = \lambda_1\Xi_{13} + \lambda_2\Xi_{23} = -\lambda_1 + \lambda_2$
- Constraint equations:
  - $\dot{z}_1 - \dot{z}_3 = 0$
  - $\dot{z}_2 + \dot{z}_3 = 0$
- We can solve this to get $\ddot{z}_1 = -\ddot{z}_2 = \ddot{z}_3 = -\frac{m_1 - m_2}{m_p + m_1 + m_2}g$
  - $\lambda_1 = \frac{m_p + 2m_2}{m_p + m_1 + m_2}m_1g$
  - $\lambda_2 = \frac{m_p + 2m_1}{-m_p + m_1 + m_2}m_2g$
  - The idea of Atwood's machine is that we can choose the masses to be nearly equal, so we get an effective g that's very small
  - λ_1 and λ_2 turn out to be the tension on the two sides of the string
- In general, the Lagrange multipliers have meaning – they are associated with the constraint forces

Lecture 16, Oct 31, 2023

Midterm Review

- Example: pendulum with spring and dashpot on the arm
  - Kinetic energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right)$
  - Potential energy: $V = \frac{1}{2}kx^2 - mg(l+x)\cos\theta$
  - Nonconservative forces: consider $\widehat{\delta W}_\triangle = Q_{x,\triangle}\delta x + Q_{\theta,\triangle}\delta\theta$
    - For a small virtual displacement δx we would have done work $f_d\delta x = -c\dot{x}\delta x$
    - A small displacement $\delta\theta$ does no non-conservative work
    - Therefore $Q_{x,\triangle} = -c\dot{x}, Q_{\theta,\triangle} = 0$

    - We could also do this using $Q_{k,\triangle} = \sum_i \underset{\rightarrow}{f}_{i,\triangle} \cdot \frac{\partial \underset{\rightarrow}{r}_i}{\partial q_k}$
  - $L = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right) - \frac{1}{2}kx^2 - mg(l+x)\cos\theta$
  - $\frac{\partial L}{\partial \dot{x}} = m\dot{x} \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$
  - $\frac{\partial L}{\partial x} = m(l+x)\dot{\theta}^2 - kx - mg\cos\theta$
  - $\frac{\partial L}{\partial \dot{\theta}} = m(l+x)^2\dot{\theta} \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2m(l+x)\dot{x}\dot{\theta} + m(l+x)^2\ddot{\theta}$
  - $\frac{\partial L}{\partial \theta} = -mg(l+x)\sin\theta$
  - $m\ddot{x} - m(l+x)\dot{\theta}^2 + kx - mg\cos\theta = -c\dot{x}$
  - $m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta} + mg(l+x)\sin\theta = 0$

Lecture 17, Nov 14, 2023

Calculus of Variations – The Brachistochrone Problem

- Consider a ball starting at point $A = (x_1, y_1)$, rolling under the influence of gravity to point $B = (x_2, y_2)$; what is the shape of the curve that minimizes the travel time?
- We want to find $y(x)$ such that $y(x_1) = y_1, y(x_2) = y_2$ and the travel time $T = \int_A^B \mathrm{d}t$ is minimized
  - Consider a differential curve element $\mathrm{d}s$; $\frac{\mathrm{d}s}{\mathrm{d}t} = v$ so $\mathrm{d}t = \frac{\mathrm{d}s}{v}$
  - $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$
  - We can find v by conservation of energy: $\frac{1}{2}mv^2 + mgy = E \implies v = \sqrt{2g(y_0 - y)}$
  - The problem becomes:
    - Minimize $\frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\sqrt{\frac{1+(y')^2}{y_0 - y}}\,\mathrm{d}x$ over $y(x)$
    - Subject to $y(x_1) = y_1, y(x_2) = y_2$
- We can generalize this to minimizing $I = \int_{x_1}^{x_2} F(y, y', x)\,\mathrm{d}x$
  - I is a *functional* – a function that takes a function and gives a number
  - How do we minimize with respect to an entire function?
- Let $y^*(x)$ be a minimum, and let $y(x) = y^*(x) + \epsilon\eta(x)$, where ϵ is small and $\eta(x)$ is any function subject to $\eta(x_1) = \eta(x_2) = 0$ (so our boundary conditions are satisfied)
  - $y' = {y^*}' + \epsilon\eta' \implies F(y, y', x) = F(y^* + \epsilon\eta, {y^*}' + \epsilon\eta', x)$
  - $I = \Phi(\epsilon) = \int_{x_1}^{x_2} F(y^* + \epsilon\eta, {y^*}' + \epsilon\eta', x)\,\mathrm{d}x$ and $\epsilon = 0$ must be a minimum, if y^* is a minimum
  - Therefore the necessary condition is $\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0$
- $$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}\epsilon} &= \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\epsilon} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}\epsilon} \\ &= \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\end{aligned}$$

- $\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) \mathrm{d}x$
  $= \left[\frac{\partial F}{\partial y'}\eta\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right] \eta \, \mathrm{d}x$
  $= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right] \eta \, \mathrm{d}x$
  * Note we applied integration by parts and used the boundary conditions $\eta(x_1) = \eta(x_2) = 0$
- Since this is always equal to 0 and η can be anything, we can conclude that the part inside the brackets must always be zero
  * This is can be proven and is known as the *Fundamental Lemma of Variational Calculus*

- Therefore the optimality condition is $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$
  - Notice that this is identical to Lagrange's equation in a conservative system
  - This is the *Euler-Lagrange Equation*
- An alternative way to derive this is to let $\delta y = y - y^*$ (which behaves like $\epsilon\eta$, then we would have $\delta I = 0$; we refer to this as a *stationary value* for I
  - δ is the *variational operator*; think of it as taking the Taylor expansion of a function
    * $\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$
    * Note derivatives, integrals, and δ commute
    * $\delta\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\delta y)$
    * $\delta \int f \, \mathrm{d}x = \int \delta f \, \mathrm{d}x$
  - $\delta I = \int_{x_1}^{x_2} \delta F(y, y', x) \, \mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right) \mathrm{d}x = 0$
  - We can then apply integration by parts and use $\delta y(x_1) = \delta y(x_2) = 0$, and apply the fundamental lemma as normal

Hamiltonian Mechanics

- *Hamilton's Principle*: the motion of a system, under the influence of conservative forces, from time t_1 to t_2, is given by the stationary value of the functional $I = \int_{t_1}^{t_2} L \, \mathrm{d}t$, where $L = T - V$
  - Under non-conservative forces, we instead have $\delta \int_{t_1}^{t_2} L \, \mathrm{d}t + \int_{t_1}^{t_2} \delta W_{\triangle} \, \mathrm{d}t = 0$, where $\delta W_{\triangle}$ is the virtual work done by non-conservative forces
  - This is the *extended Hamilton's principle*
- Hamilton's principle, like Newton's laws and Lagrange's principle, is an equivalent description of classical mechanics; all 3 apply at each instance in time
  - Unlike the other two however, Hamiltonian mechanics looks at an interval of time, while the other two methods look at an instant in time
- The *Hamiltonian* is defined by $H = \sum_k \dot{q}_k p_k - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is the generalized momentum of each coordinate
  - $\mathrm{d}H = \sum_k \left(p_k \, \mathrm{d}\dot{q}_k + \dot{q}_k \, \mathrm{d}p_k - \frac{\partial L}{\partial q_k} \mathrm{d}q_k - \frac{\partial L}{\partial \dot{q}_k} \mathrm{d}\dot{q}_k\right) - \frac{\partial L}{\partial t} \mathrm{d}t$
  - For a conservative system, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \implies \dot{p}_k = \frac{\partial L}{\partial q_k}$
  - $\mathrm{d}H = \sum_k (\dot{q}_k \, \mathrm{d}p_k - \dot{p}_k \, \mathrm{d}q_k) - \frac{\partial L}{\partial t} \mathrm{d}t$
    * We have shown that any differential in H is given by differentials in $\boldsymbol{p}$, $\boldsymbol{q}$ and t
    * Therefore $H = H(\boldsymbol{q}, \boldsymbol{p}, t)$ and has no dependence on $\dot{q}$

- By the chain rule, $\mathrm{d}H = \sum_k \left(\frac{\partial H}{\partial p_k}\,\mathrm{d}p_k + \frac{\partial H}{\partial q_k}\,\mathrm{d}q_k\right) + \frac{\partial H}{\partial t}\,\mathrm{d}t$
    - Therefore $\dot{q}_k = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k}$
    - These are known as *Hamilton's Canonical Equations*, and can serve as an alternative formulation of mechanics

Lecture 18, Nov 16, 2023

Dynamics of Rigid Bodies

Momentum of Rigid Bodies

- Consider a rigid body R and some inertial reference point $O_{\mathfrak{I}}$
    - Each differential mass element $\mathrm{d}m$ has momentum $\mathrm{d}\underset{\to}{p} = \underset{\to}{r}^{\bullet}\,\mathrm{d}m$
    - Therefore the overall momentum is $\underset{\to}{p} = \int_R \underset{\to}{r}^{\bullet}\,\mathrm{d}m$
- Consider some reference point O fixed to the body, and let $\underset{\to}{\rho}$ be the position of a mass element relative to O
    - $\underset{\to}{r}^{\bullet} = \underset{\to}{v}_O + \underset{\to}{\rho}^{\bullet} = \underset{\to}{v}_O + \underset{\to}{\rho}^{\circ} + \underset{\to}{\omega} \times \underset{\to}{\rho}$
    - But $\underset{\to}{\rho}$ is fixed as seen in a body-relative frame, so $\underset{\to}{\rho}^{\circ} = 0$ (unless the body is deformable)
    - $\underset{\to}{p} = \int_R (\underset{\to}{v}_O + \underset{\to}{\omega} \times \underset{\to}{\rho})\,\mathrm{d}m = \int_R \underset{\to}{v}_O\,\mathrm{d}m - \int_R \underset{\to}{\rho} \times \underset{\to}{\omega}\,\mathrm{d}m = m\underset{\to}{v}_O - \left(\int_R \underset{\to}{\rho}\,\mathrm{d}m\right) \times \underset{\to}{\omega}$
- Let $\underset{\to}{c}_O = \int_R \underset{\to}{\rho}\,\mathrm{d}m$ be the *first moment of mass* (aka *first moment of inertia*), then $\underset{\to}{p} = m\underset{\to}{v}_O - \underset{\to}{c}_O \times \underset{\to}{\omega}$
    - Note that $\underset{\to}{c}_O$ has a subscript since it is computed with respect to O
    - Expressed in a body frame $\underset{\to}{\mathcal{F}}_b$, $\boldsymbol{p} = m\boldsymbol{v}_O - \boldsymbol{c}_O^{\times}\boldsymbol{\omega}$
        - Note $\boldsymbol{c}_O$ is a constant in $\underset{\to}{\mathcal{F}}_b$
- For angular momentum, $\mathrm{d}\underset{\to}{h}_O = \underset{\to}{\rho} \times \mathrm{d}\underset{\to}{p} = \underset{\to}{\rho} \times (\underset{\to}{v}_O - \underset{\to}{\rho} \times \underset{\to}{\omega})\,\mathrm{d}m \implies \underset{\to}{h}_O = \underset{\to}{c}_O \times \underset{\to}{v}_O - \int_R \underset{\to}{\rho} \times (\underset{\to}{\rho} \times \underset{\to}{\omega})\,\mathrm{d}m$
    - In $\underset{\to}{\mathcal{F}}_b$,
    $$\begin{aligned}\boldsymbol{h}_O &= \boldsymbol{c}_O^{\times}\boldsymbol{v}_O - \int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\boldsymbol{\omega}\,\mathrm{d}m \\ &= \boldsymbol{c}_O^{\times}\boldsymbol{v}_O + \left(-\int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\,\mathrm{d}m\right)\boldsymbol{\omega} \\ &= \boldsymbol{c}_O^{\times}\boldsymbol{v}_O + \boldsymbol{J}_O\boldsymbol{\omega}\end{aligned}$$
    - $\boldsymbol{J}_O = -\int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\,\mathrm{d}m$ is the *second moment of mass*, or the *inertia matrix*
        - Note $\underset{\to}{J}_O$ is a *second order tensor*; in vector form, $\underset{\to}{h}_O = \underset{\to}{c}_O \times \underset{\to}{v}_O + \underset{\to}{J}_O \cdot \underset{\to}{\omega}$
        - $\boldsymbol{J}_O = \int_R (\rho^2\mathbf{1} - \boldsymbol{\rho}\boldsymbol{\rho}^T)\,\mathrm{d}m$
    - Let $\boldsymbol{s}$ be any vector, then
    $$\begin{aligned}\boldsymbol{s}^T\boldsymbol{J}_O\boldsymbol{s} &= -\boldsymbol{s}^T\left(\int_R \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\,\mathrm{d}m\right)\boldsymbol{s} \\ &= -\int_R \boldsymbol{s}^T\boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\boldsymbol{s}\,\mathrm{d}m \\ &= \int_R (\boldsymbol{\rho}^{\times}\boldsymbol{s})^T(\boldsymbol{\rho}^{\times}\boldsymbol{s})\,\mathrm{d}m \\ &= \int_R \|\boldsymbol{\rho}^{\times}\boldsymbol{s}\|^2\,\mathrm{d}m\end{aligned}$$
        - There will always be some $\underset{\to}{\rho}$ that is not parallel to $\underset{\to}{s}$ for any 3D body, so this integral is always positive for a nonzero $\underset{\to}{s}$
        - Therefore $\boldsymbol{J}_O$ is symmetric positive definite (hence why we include a minus sign in the definition)
- A *second-order tensor* $\underset{\to}{D} = \underset{\to}{a}\underset{\to}{b}$ is defined such that $\underset{\to}{D} \cdot \underset{\to}{v} = (\underset{\to}{a}\underset{\to}{b}) \cdot \underset{\to}{v} = \underset{\to}{a}(\underset{\to}{b} \cdot \underset{\to}{v})$

- In matrix form, $\begin{bmatrix}\boldsymbol{p}\\ \boldsymbol{h}_O\end{bmatrix} = \begin{bmatrix}m\mathbf{1} & -\boldsymbol{c}_O^\times\\ \boldsymbol{c}_O^\times & \boldsymbol{J}_O\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_O\\ \boldsymbol{\omega}\end{bmatrix}$, where $\boldsymbol{M} = \begin{bmatrix}m\mathbf{1} & -\boldsymbol{c}_O^\times\\ \boldsymbol{c}_O^\times & \boldsymbol{J}_O\end{bmatrix}$ is the *mass matrix*, which is also symmetric positive definite
- If we choose $O = ◕$, then $\int_R \underset{\rightarrow}{r}\,\mathrm{d}m = 0$, so $\underset{\rightarrow}{c} = 0$
  - $\boldsymbol{p} = m\boldsymbol{v}_O$
  - $\boldsymbol{h}_{◕} = \boldsymbol{J}_{◕}\boldsymbol{\omega} = \boldsymbol{I}\boldsymbol{\omega}$, where we denote $\boldsymbol{I}$ as the inertia matrix about the centre of mass
- Consider two inertia matrices $\boldsymbol{J}_A, \boldsymbol{J}_B$ relative to points A, B; for a differential mass element, denote position relative to A by $\underset{\rightarrow}{a}$, position relative to B by $\underset{\rightarrow}{b}$ and the relative position between A and B is $\underset{\rightarrow}{\rho}^{BA}$
  - $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{\rho}^{BA}$ in a common body frame (draw this out)
  - $\boldsymbol{J}_A = -\int_R \boldsymbol{a}^\times\boldsymbol{a}^\times\,\mathrm{d}m$
    $= -\int_R (\boldsymbol{b} + \boldsymbol{\rho}^{BA})^\times(\boldsymbol{b} + \boldsymbol{\rho}^{BA})\,\mathrm{d}m$
    $= -\int_R \left(\boldsymbol{b}^\times\boldsymbol{b}^\times + \boldsymbol{\rho}^{BA^\times}\boldsymbol{b}^\times + \boldsymbol{b}^\times\boldsymbol{\rho}^{BA^\times} + \boldsymbol{\rho}^{BA^\times}\boldsymbol{\rho}^{BA^\times}\right)\mathrm{d}m$
    $= -\int_R \boldsymbol{b}^\times\boldsymbol{b}^\times\,\mathrm{d}m - \boldsymbol{\rho}^{BA^\times}\int_R \boldsymbol{b}^\times\,\mathrm{d}m - \int_R \boldsymbol{b}^\times\,\mathrm{d}m\boldsymbol{\rho}^{BA^\times} - \int \mathrm{d}m\boldsymbol{\rho}^{BA^\times}\boldsymbol{\rho}^{BA^\times}$
    $= \boldsymbol{J}_B - \boldsymbol{\rho}^{BA^\times}\boldsymbol{c}_B^\times - \boldsymbol{c}_B^\times\boldsymbol{\rho}^{BA^\times} - m\boldsymbol{\rho}^{BA^\times}\boldsymbol{\rho}^{BA^\times}$
  - This is the *parallel axis theorem* for an inertia matrix

Theorem

Parallel Axis Theorem: Given an inertia matrix $\boldsymbol{J}_B$ around a point B, and relative position $\boldsymbol{\rho}^{BA}$ from A to B, we can find the inertia matrix around A, $\boldsymbol{J}_A$ as:

$$\boldsymbol{J}_A = \boldsymbol{J}_B - \boldsymbol{\rho}^{BA^\times}\boldsymbol{c}_B^\times - \boldsymbol{c}_B^\times\boldsymbol{\rho}^{BA^\times} - m\boldsymbol{\rho}^{BA^\times}\boldsymbol{\rho}^{BA^\times}$$

- Consider the same reference point in two frames $\underset{\rightarrow}{\mathcal{F}}_a, \underset{\rightarrow}{\mathcal{F}}_b$; denote $\boldsymbol{J}_a, \boldsymbol{J}_b$ be the inertia matrix about this point expressed in the two frames
  - $\boldsymbol{J}_a = -\int_R \boldsymbol{\rho}_a^\times\boldsymbol{\rho}_a^\times\,\mathrm{d}m$
    $= -\int_R (\boldsymbol{C}_{ab}\boldsymbol{\rho}_b)^\times(\boldsymbol{C}_{ab}\boldsymbol{\rho}_b)^\times\,\mathrm{d}m$
    $= -\int_R (\boldsymbol{C}_{ab}\boldsymbol{\rho}_b^\times\boldsymbol{C}_{ba})(\boldsymbol{C}_{ab}\boldsymbol{\rho}_b^\times\boldsymbol{C}_{ba})\,\mathrm{d}m$
    $= \boldsymbol{C}_{ab}\left(-\int_R \boldsymbol{\rho}_b^\times\boldsymbol{\rho}_b^\times\right)\boldsymbol{C}_{ba}$
    $= \boldsymbol{C}_{ab}\boldsymbol{J}_b\boldsymbol{C}_{ba}$
  - This is the *rotational transformation theorem* for an inertia matrix
  - Note for a second-order tensor, $\underset{\rightarrow}{J} = \underset{\rightarrow}{\mathcal{F}}_a^T\boldsymbol{J}_a\underset{\rightarrow}{\mathcal{F}}_a \iff \boldsymbol{J}_a = \underset{\rightarrow}{\mathcal{F}}_a \cdot \underset{\rightarrow}{J} \cdot \underset{\rightarrow}{\mathcal{F}}_a^T$, so this identity follows
    * The result applies for any second-order tensor

Motion of Rigid Bodies

- To get the equations of motion, we can treat a rigid body like a grammar of particles
- For a grammar of particles, $\underset{\rightarrow}{p}^{\bullet} = \underset{\rightarrow}{f}$ and $\underset{\rightarrow}{h}_O^{\bullet} + \underset{\rightarrow}{v}_O \times \underset{\rightarrow}{p} = \underset{\rightarrow}{\tau}_O$
  - $\boldsymbol{p} = m\boldsymbol{v}_O - \boldsymbol{c}_O^\times\boldsymbol{\omega}$
  - $\boldsymbol{h}_O = \boldsymbol{c}_O^\times\boldsymbol{v}_O + \boldsymbol{J}_O\boldsymbol{\omega}$
  - But to use these, we have to first convert the derivative $(\cdot)^{\bullet}$ with respect to inertial frame into a derivative with respect to body frame

    * $\underrightarrow{p}^{\circ} + \underrightarrow{\omega} \times \underrightarrow{p} = \underrightarrow{f}$
    * $\underrightarrow{h}_O^{\circ} + \underrightarrow{\omega} \times \underrightarrow{h}_O + \underrightarrow{v}_O \times \underrightarrow{p} = \underrightarrow{\tau}_O$
  - In the body frame:
    * $\dot{\boldsymbol{p}} + \boldsymbol{\omega}^\times \boldsymbol{p} = \boldsymbol{f}$
    * $\dot{\boldsymbol{h}}_O + \boldsymbol{\omega}^\times \boldsymbol{h}_O + \boldsymbol{v}_O^\times \boldsymbol{p} = \boldsymbol{\tau}_O$
- Therefore the equations of motion for a rigid body are given by, in the general case:
  - $m\dot{\boldsymbol{v}}_O - \boldsymbol{c}_O^\times \dot{\boldsymbol{\omega}} + m\boldsymbol{\omega}^\times \boldsymbol{v}_O - \boldsymbol{\omega}^\times \boldsymbol{c}_O^\times \boldsymbol{v}_O = \boldsymbol{f}$
  - $\boldsymbol{c}_O^\times \dot{\boldsymbol{v}}_O + \boldsymbol{J}_O \dot{\boldsymbol{\omega}} - \boldsymbol{c}_O^\times \boldsymbol{\omega}^\times \boldsymbol{v}_O + \boldsymbol{\omega}^\times \boldsymbol{J}_O \boldsymbol{\omega} = \boldsymbol{\tau}_O$
  - In matrix form, $\boldsymbol{M} \begin{bmatrix} \dot{\boldsymbol{v}}_O \\ \dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}^\times & \boldsymbol{0} \\ \boldsymbol{v}_O^\times & \boldsymbol{\omega}^\times \end{bmatrix} \boldsymbol{M} \begin{bmatrix} \boldsymbol{v}_O \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{\tau}_O \end{bmatrix}$
    * $\begin{bmatrix} \boldsymbol{v}_O \\ \boldsymbol{\omega} \end{bmatrix}$ is a generalized velocity and $\begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{\tau}_O \end{bmatrix}$ is a generalized force
  - If $O =$ ◕, we can simplify (where all quantities are relative to the centre of mass):
    * $m\dot{\boldsymbol{v}} + m\boldsymbol{\omega}^\times \boldsymbol{v} = \boldsymbol{f}$
    * $\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}$
  - Note that in the general case, the equations of motion are coupled; but if we use ◕, the rotational equation is uncoupled, making it much easier to solve
  - Solving this gives us the angular velocity of the body, but not the orientation; for that we need to use Poisson's equation $\dot{\boldsymbol{C}} + \boldsymbol{\omega}^\times \boldsymbol{C} = \boldsymbol{0}$, or for Euler angles $\boldsymbol{\omega} = \boldsymbol{S}\dot{\boldsymbol{\theta}}$ (or axis-angle/quaternion)
- Kinetic energy: $$\begin{aligned} T &= \frac{1}{2}\int_R \underrightarrow{r}^{\bullet} \cdot \underrightarrow{r}^{\bullet} \,\mathrm{d}m \\ &= \frac{1}{2}\int_R (\underrightarrow{v}_O - \underrightarrow{\rho} \times \underrightarrow{\omega}) \cdot (\underrightarrow{v}_O - \underrightarrow{\rho} \times \underrightarrow{\omega}) \,\mathrm{d}m \\ &= \frac{1}{2}\int_R (\boldsymbol{v}_O - \boldsymbol{\rho}^\times \boldsymbol{\omega})^T (\boldsymbol{v}_O - \boldsymbol{\rho}^\times \boldsymbol{\omega}) \,\mathrm{d}m \\ &= \frac{1}{2} m \boldsymbol{v}_O^T \boldsymbol{v}_O - \boldsymbol{v}_O^T \boldsymbol{c}_O^\times \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{J}_O \boldsymbol{\omega} \end{aligned}$$
  - Notice that this has 2 parts: translational, rotational, and a coupling term, which disappears when we use the centre of mass reference frame
  - In matrix form, $\boldsymbol{T} = \frac{1}{2}\begin{bmatrix} \boldsymbol{v}_O \\ \boldsymbol{\omega} \end{bmatrix}^T \boldsymbol{M} \begin{bmatrix} \boldsymbol{v}_O \\ \boldsymbol{\omega} \end{bmatrix}$
- Note we can expand the inertia matrix as $\boldsymbol{I} = \int_R \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \mathrm{d}m$
  - In general $\boldsymbol{I}$ will be fully populated; but can we diagonalize it?
  - We know $\boldsymbol{I}$ is symmetric positive definite, so it is diagonalizable and the eigenvector matrix is orthogonal
  - There will always be a $\boldsymbol{E}$ such that $\boldsymbol{E}^{-1}\boldsymbol{I}\boldsymbol{E} = \boldsymbol{\Lambda}$; but since $\boldsymbol{E}^{-1} = \boldsymbol{E}^T$ (and choose $\boldsymbol{E}$ such that $\det \boldsymbol{E} = 1$), it is a rotation matrix
  - Recall that the inertia matrix transforms as $\boldsymbol{C}_{ab}\boldsymbol{I}_b\boldsymbol{C}_{ba} = \boldsymbol{I}_a$, which has the exact same form as the diagonalization we found
  - Therefore we can always find a reference frame such that $\boldsymbol{I}$ is diagonal; this is referred to as the *principal-axis frame*
  - This works even if we don't use ◕ as our reference point
- In the principal axis frame, with ◕ as our reference point, the rotational equation reduces to:
  - $I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1$
  - $I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \tau_2$
  - $I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3$
  - These are known as *Euler's equations*

Summary

For a rigid body, let $\boldsymbol{\rho}$ be the position of a differential mass element relative to O in a body-fixed frame, then:

- The *first moment of mass/inertia* $\boldsymbol{c}_O = \int_R \boldsymbol{\rho}\, \mathrm{d}m$, which is zero if $O =$ ◕
- The *second moment of mass/inertia matrix* $\boldsymbol{J}_O = -\int_R \boldsymbol{\rho}^\times \boldsymbol{\rho}^\times\, \mathrm{d}m$, which is diagonal in the principal axis frame and denoted $\boldsymbol{I}$ if $O =$ ◕

Then the linear and angular momenta are given by

$$\boldsymbol{p} = m\boldsymbol{v}_O - \boldsymbol{c}_O^\times \boldsymbol{\omega}, \quad \boldsymbol{h}_O = \boldsymbol{c}_O^\times \boldsymbol{v}_O + \boldsymbol{J}_O \boldsymbol{\omega}$$

The equations of motion are given by, in general,

$$\begin{gathered} m\dot{\boldsymbol{v}}_O - \boldsymbol{c}_O^\times \dot{\boldsymbol{\omega}} + m\boldsymbol{\omega}^\times \boldsymbol{v}_O - \boldsymbol{\omega}^\times \boldsymbol{c}_O^\times \boldsymbol{v}_O = \boldsymbol{f} \\ \boldsymbol{c}_O^\times \dot{\boldsymbol{v}}_O + \boldsymbol{J}_O \dot{\boldsymbol{\omega}} - \boldsymbol{c}_O^\times \boldsymbol{\omega}^\times \boldsymbol{v}_O + \boldsymbol{\omega}^\times \boldsymbol{J}_O \boldsymbol{\omega} = \boldsymbol{\tau}_O \end{gathered}$$

Using $O =$ ◕, this reduces to

$$\begin{gathered} m\dot{\boldsymbol{v}} + m\boldsymbol{\omega}^\times \boldsymbol{v} = \boldsymbol{f} \\ \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau} \end{gathered}$$

And the kinetic energy is given by, in general

$$T = \frac{1}{2} m \boldsymbol{v}_O^T \boldsymbol{v}_O - \boldsymbol{v}_O^T \boldsymbol{c}_O^\times \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{J}_O \boldsymbol{\omega}$$

where the middle coupling term disappears when using $O =$ ◕.

Lecture 19, Nov 21, 2023

Spin Stability of Rigid Bodies

- Consider a system with no external torque, spinning at a constant nominal rate $\boldsymbol{\omega} = \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix}$; under what conditions is this system stable?
- Consider a small perturbation, such that $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega}(t)$ where $\Delta\boldsymbol{\omega}(t) = \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \end{bmatrix}$
- Plugging into Euler's equations: $\begin{cases} I_1 \Delta\dot{\omega}_1 - (I_2 - I_3)(\nu + \Delta\omega_1)\Delta\omega_3 = 0 \\ I_2 \Delta\dot{\omega}_2 - (I_3 - I_1)\Delta\omega_3 \Delta\omega_1 = 0 \\ I_3 \Delta\dot{\omega}_3 - (I_1 - I_2)(\nu + \Delta\omega_2) = 0 \end{cases}$
  - After linearizing: $\begin{cases} I_1 \Delta\dot{\omega}_1 - (I_2 - I_3)\nu\Delta\omega_3 = 0 \\ I_2 \Delta\dot{\omega}_2 = 0 \\ I_3 \Delta\dot{\omega}_3 - (I_1 - I_2)\nu\Delta\omega_1 = 0 \end{cases}$
  - $\Delta\omega_2$ is then constant, so it is always stable
  - Taking the derivative of the first equation and substituting in $\Delta\dot{\omega}_3$, we can get a differential equation for $\Delta\omega_1$
  - $\Delta\ddot{\omega}_1 + \dfrac{(I_2 - I_3)(I_2 - I_1)}{I_1 I_3}\nu^2 \Delta\omega_1 = 0 \implies \Delta\ddot{\omega}_1 + \beta^2 \Delta\omega_1 = 0$
    - Note we could do the same to the second equation and we would get something in the exact same form, with the same β^2

- This is now an oscillator, so for stability we need $\beta^2 > 0$; $\beta^2 \leq 0$ makes it unstable

- For $\beta^2 > 0$ we need $I_2 - I_3$ and $I_2 - I_1$ to have the same sign, for a rotation about the 2 axis to be stable
  - This requires either $I_2 > I_1, I_3$ or $I_2 < I_1, I_3$ – it has to be the major (largest inertia) or minor (smallest inertia) axis, but not the intermediate axis
- From the equation of motion: $\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \boldsymbol{I}\boldsymbol{\omega} = 0 \implies \boldsymbol{\omega}^T\boldsymbol{I}\dot{\boldsymbol{\omega}} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega}\right) = 0$
  - Note the $\boldsymbol{\omega}^T\boldsymbol{\omega}^\times$ cancels
  - Integrating this, we get $\frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega} = T$ is a constant – this is the rotational kinetic energy
  - In the principal axis frame, expanding this out we get $\frac{\omega_1^2}{\frac{2T}{I_1}} + \frac{\omega_2^2}{\frac{2T}{I_2}} + \frac{\omega_3^2}{\frac{2T}{I_3}} = 0$
  - Geometrically, means that $\boldsymbol{\omega}$ must lie on the surface of an ellipsoid – the energy ellipsoid
- Multiplying instead by $\boldsymbol{\omega}^T\boldsymbol{I}$, we have $\boldsymbol{\omega}^T\boldsymbol{I}^2\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega}^\times\boldsymbol{I}\boldsymbol{\omega} = 0 \implies \boldsymbol{\omega}^T\boldsymbol{I}^2\dot{\boldsymbol{\omega}} = 0$
  - Note $\boldsymbol{z}^T\boldsymbol{\omega}^\times\boldsymbol{z} = 0$ for any skew-symmetric $\boldsymbol{\omega}^\times$ (since it is a scalar, and if you transpose it you get its negative)
  - Doing the same and integrating gives us $\boldsymbol{\omega}^T\boldsymbol{I}^2\boldsymbol{\omega} = h^2$, another constant – this is the square of the angular momentum
  - $\frac{\omega_1^2}{\frac{h^2}{I_1^2}} + \frac{\omega_2^2}{\frac{h^2}{I_2^2}} + \frac{\omega_3^2}{\frac{h^2}{I_3^2}} = 0$
  - Geometrically this gives us yet another ellipsoid for $\boldsymbol{\omega}$ – the momentum ellipsoid

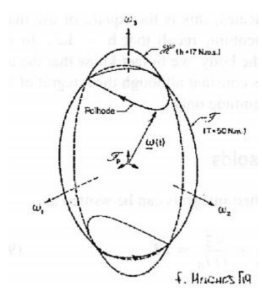

Figure 7: Intersection of the energy and momentum ellipsoids.

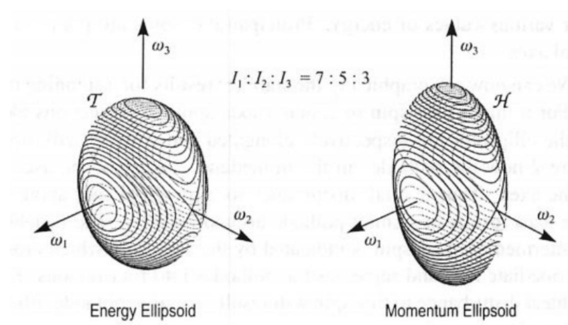

Figure 8: The energy and momentum ellipsoids with polhodes.

- Since $\boldsymbol{\omega}$ has to be on both ellipsoids, it must stay on their intersection – these intersection lines are called *polhodes*
  - Due to the squaring of I, the momentum ellipsoid will usually appear more stretched out than the energy ellipsoid
  - Specifying the energy and angular momentum initial conditions choose a pair of polhodes; a third initial condition is needed to solve for the angular momentum as a function of time
  - For a pure spin about the minor axis, the momentum ellipsoid is entirely contained within the

    energy ellipsoid, so the only intersections are the top and bottom points
        * With a small perturbation, the momentum ellipsoid increases slightly in size, so we get a polhode near the pole, which is a small circle; since the angular momentum stays within the circle, this means we are stable
    - For a pure spin about the major axis, the energy ellipsoid is entirely contained within the momentum ellipsoid
- Notice $T = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega}$ so $\boldsymbol{\nabla}_\omega T = \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{h}$
    - Therefore $\boldsymbol{h}$ is always normal to the energy ellipsoid, and moreover it is fixed in inertial space since there are no external torques
    - We can interpret this as the energy ellipsoid "rolling" on the *invariable plane*
        * This is because $T = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{h}$ is constant, so the projection of $\underrightarrow{\omega}$ onto $\underrightarrow{h}$ is constant – the tip of $\underrightarrow{\omega}$ must lie on a plane normal to $\underrightarrow{h}$
    - As we roll the ellipsoid, $\boldsymbol{\omega}$ remains on both the surface of the energy ellipsoid and the invariable plane
    - This is known as *Poinsot's Geometric Interpretation*
    - The curve traced out on the invariable plane is known as the *herpolhode*
    - Since the energy ellipsoid exists in the principal axis frame, which is a body-fixed frame, the motion of the energy ellipsoid is the motion of the body

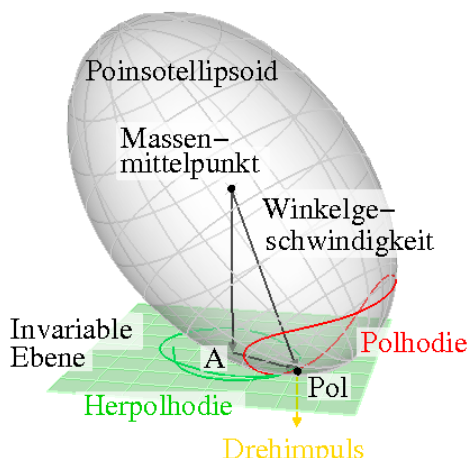

Figure 9: Poinsot's construction, in German for some reason.

- In real life however, since any body dissipates energy, $\dot{T} < 0$, so a minor axis spin will slowly shift towards a major axis spin; as long as the system can lose energy, minor axis spins are unstable
    - The major axis spin is asymptotically stable since the energy and momentum ellipsoids must intersect, so at this point the energy ellipsoid can't spin more

Lecture 20, Nov 23, 2023

Analysis of a Spinning Top

- Consider a general spinning top spinning about its axis of symmetry, with the contact point fixed; what is the rate of the precession of $\underrightarrow{h}$ about the vertical axis?
- In general we have 2 contributors to $\underrightarrow{h}$: the spin of the top itself and the wobbling; we will assume that the top is spinning fast enough that almost all $\underrightarrow{h}$ lies in the spinning of the top itself
- We also assume the angular velocity has constant magnitude
- We want $\frac{\mathrm{d}\phi}{\mathrm{d}t}$, where $\mathrm{d}\phi$ is a small change to the angle of the projection of $\underrightarrow{h}$ onto the horizontal plane

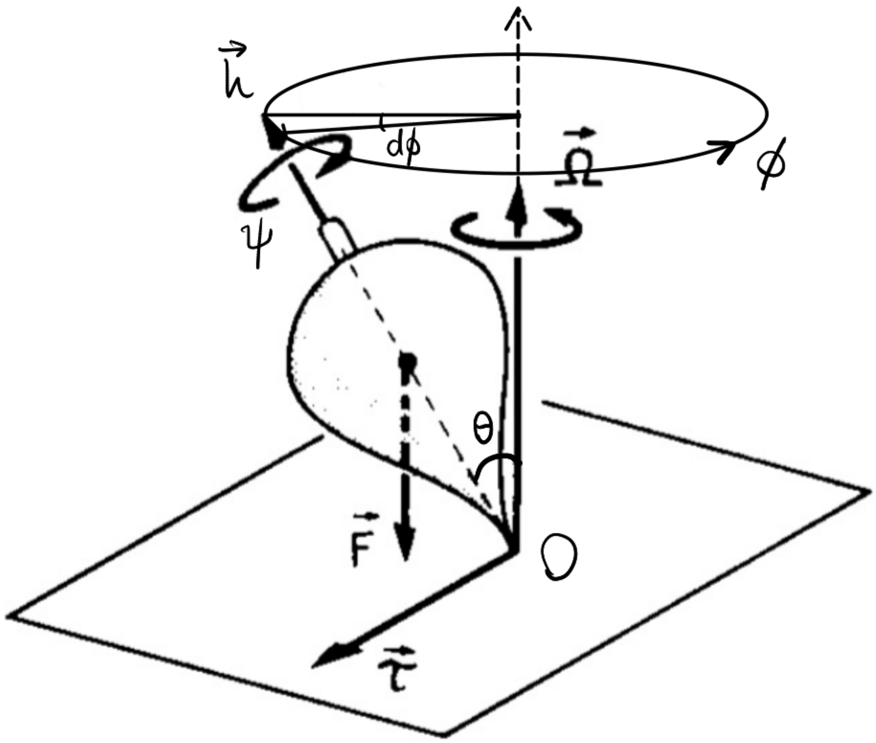

Figure 10: Precession of a spinning top.

  - $\mathrm{d}\phi = \dfrac{\mathrm{d}h}{h\sin\theta}$ where θ is the angle made with the vertical axis (note θ is called the *nutation angle*)
- $h = J_a\nu$ where ν is the spin rate and J_a is the axial moment of inertia
- The change in $\underset{\to}{h}$ is due to the only externally applied force – gravity
  - Gravity acts at the centre of mass, which is a distance r from O
  - Therefore it exerts a torque $\underset{\to}{\tau} = m\underset{\to}{g}r\sin\theta = \dot{\underset{\to}{h}}$
  - The magnitude is then $\tau = mgr\sin\theta = \dfrac{\mathrm{d}h}{\mathrm{d}t}$
- Substitute relevant quantities: $\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = \dfrac{\frac{\mathrm{d}h}{\mathrm{d}t}}{J_a\nu\sin\theta} = \dfrac{mgr\sin\theta}{J_a\nu\sin\theta} = \dfrac{mgr}{J_a\nu}$
  - This is the *precession rate*
- To analyze the full motion, we will use the Lagrangian formulation; consider a 3-1-3 set of Euler angles ϕ, θ, ψ where the 3-axis is the vertical axis
  - Recall: $\boldsymbol{\omega} = \boldsymbol{S}(\theta,\psi)\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}\dot{\phi}\sin\theta\sin\phi + \dot{\theta}\cos\psi\\ \dot{\theta}\sin\theta\cos\psi - \theta\sin\psi\\ \dot{\phi}\cos\theta + \dot{\psi}\end{bmatrix}$
  - The kinetic energy is $T = \dfrac{1}{2}\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega}$, where $\boldsymbol{J} = \begin{bmatrix}J_t & & \\ & J_t & \\ & & J_a\end{bmatrix}$ (assuming symmetry)
    - We will call the transverse moments of inertial J_t and the axial one J_a
  - Expanding this out: $T = \dfrac{1}{2}J_t(\dot{\theta}^2 + \phi^2\sin^2\theta) + \dfrac{1}{2}J_a(\dot{\phi}\cos\theta + \dot{\psi})^2$
  - The potential energy is taken at the centre of mass and so $V = mgr\cos\theta$
- Notice L is not dependent on ϕ and ψ, so $\dfrac{\partial L}{\partial\dot{\phi}}$ and $\dfrac{\partial L}{\partial\dot{\psi}}$ are constant (these are the angular momenta about the two axes)
  - Note these are called *cyclic* or *ignorable* coordinates

- $\dfrac{\partial L}{\partial \dot{\phi}} = J_t\dot{\phi}\sin^2\theta + J_a\cos\theta(\dot{\phi}\cos\theta + \dot{\psi}) = J_t\omega_\phi$
- $\dfrac{\partial L}{\partial \dot{\psi}} = J_a(\dot{\phi}\cos\theta + \dot{\psi}) = J_a\nu$
- With the assumption that $\dot{\phi} \ll \dot{\psi}$ we can write the first equation as $J_t\dot{\phi}\sin^2\theta + J_a\nu\cos\theta = J_t\omega_\phi$

- Finally the equation in θ: $J_t\ddot{\theta} + (J_a - J_t)\phi^2\sin\theta\cos\theta + J_a\dot{\phi}\dot{\psi}\sin\theta - mgr\sin\theta = 0$
  - We see that even with no non-conservative forces, the nutation angle still changes
  - The faster that the top is spinning, the less the nutation changes, which is why normally it seems almost constant to us

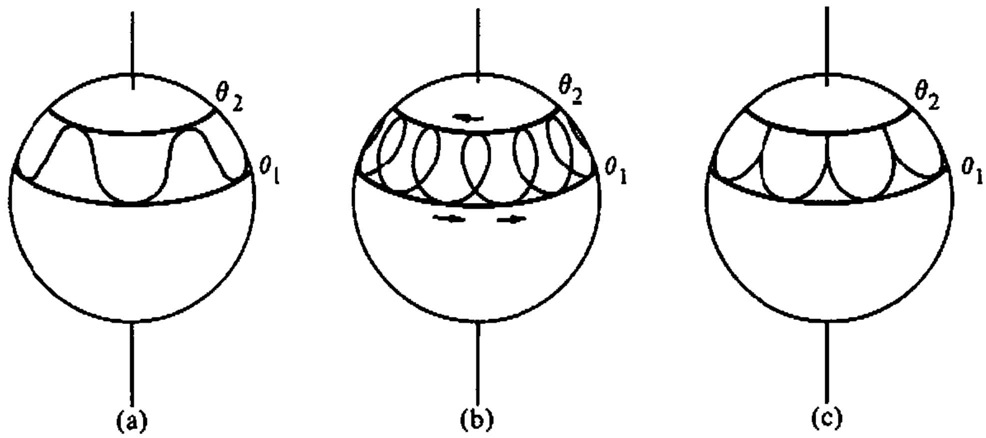

Figure 11: Types of precession of the axis of the top.

Lecture 21, Nov 28, 2023

Vibrations: Equation of Motion

- Consider a system of N rigid bodies and particles, described by a set of n generalized coordinates q_k
- Consider a potential V, then the forces are given by $f = -\dfrac{\partial V}{\partial q_k}$, which are zero at equilibrium
- WLOG choose the equilibria to be when $q_k = 0$, then we can expand the potential about the equilibrium:
  - $V(\boldsymbol{q}) = V_0 + \sum_k \dfrac{\partial V}{\partial q_k}q_k + \dfrac{1}{2}\sum_{k,j}\dfrac{\partial^2 V}{\partial q_j \partial q_k}q_j q_k$
  - We can take $V_0 = 0$ since in general the reference potential level does not matter; at an equilibrium we also have $\dfrac{\partial V}{\partial q_k} = 0$
  - Therefore $V = \dfrac{1}{2}\sum_{k,j}\dfrac{\partial^2 V}{\partial q_j \partial q_k}q_j q_k$
- We may express $V = \dfrac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}$ for small disturbances
  - $\boldsymbol{K}$ is a matrix of second partials, known as the *stiffness matrix*
  - Due to symmetry of second partials, $\boldsymbol{K}$ is symmetric (but note it is not necessarily definite)
- For kinetic energy, $T = \dfrac{1}{2}\sum_{i=1}^{N}\left(m_i\boldsymbol{v}_i^T\boldsymbol{v}_i + \boldsymbol{\omega}_i^T\boldsymbol{I}_i\boldsymbol{\omega}_i\right)$
  - $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i(q_1, \dots, q_k) = \sum_{k=1}^{n}\dfrac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k = \sum_{k=1}^{n}\boldsymbol{a}_{ik}\dot{q}_k$
  - $\boldsymbol{\omega}_i^\times = -\dot{\boldsymbol{C}}_i\boldsymbol{C}_i^T = \sum_k -\dfrac{\partial \boldsymbol{C}_i}{\partial q_k}\boldsymbol{C}_i^T\dot{q}_k = \sum_k \boldsymbol{b}_{ik}^\times\dot{q}_k \implies \boldsymbol{\omega}_i = \sum_k \boldsymbol{b}_{ik}\dot{q}_k$
  - We will assume that both have no dependence on $\boldsymbol{q}$

- Therefore $T = \frac{1}{2}\sum_{j,k}\left[\sum_i m_i \boldsymbol{a}_{ij}^T \boldsymbol{a}_{ik}\dot{q}_j\dot{q}_k + \sum_i \boldsymbol{b}_{ij}^T \boldsymbol{I}_i \boldsymbol{b}_{ik}\dot{q}_j\dot{q}_k\right]$
- Let $M_{jk} = \sum_i m_i \boldsymbol{a}_{ij}^T \boldsymbol{a}_{ij} + \sum_i \boldsymbol{b}_{ij}^T \boldsymbol{I}\boldsymbol{b}_{ij}$, then $T = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$
  - $\boldsymbol{M}$ is symmetric and positive definite, because for any nonzero $\dot{q}$, we expect some kind of positive kinetic energy
- The non-conservative forces are $\widehat{\delta W}_\triangle = \sum_k f_k \delta q_k = \delta \boldsymbol{q}_k^T \boldsymbol{f}$
- Using Hamilton's principle, we seek to find $\delta \int_{t_1}^{t_2} L\,\mathrm{d}t - \int_{t_1}^{t_2} \widehat{\delta W}_\triangle\,\mathrm{d}t = 0$
  - $$\begin{aligned}\delta \int_{t_1}^{t_2} L\,\mathrm{d}t - \int_{t_1}^{t_2} \widehat{\delta W}_\triangle\,\mathrm{d}t &= \int_{t_1}^{t_2}\left[\delta\left(\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} - \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}\right) + \widehat{\delta W}_\triangle\right]\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\left[\left(\delta\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} - \delta\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}\right) + \delta\boldsymbol{q}^T\boldsymbol{f}\right]\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\left(-\delta\boldsymbol{q}^T\boldsymbol{M}\ddot{\boldsymbol{q}} - \delta\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q} + \delta\boldsymbol{q}^T\boldsymbol{f}\right)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\delta\boldsymbol{q}^T\left(-\boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{K}\boldsymbol{q} + \boldsymbol{f}\right)\,\mathrm{d}t\end{aligned}$$
  - Note we used integration by parts and eliminated the boundary term as in the derivation for the Euler-Lagrange equation
  - Setting this to zero, we get that the term inside the brackets must be zero
- Therefore the equation of motion is $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f}(t)$
  - Notice the similarity to the 1 dimensional spring-mass system $m\ddot{x} + kx = f$
  - If we had linear damping, we could add a $\boldsymbol{D}\dot{q}$ term, where $\boldsymbol{D}$ is symmetric and positive semi-definite
  - We could also add a $\boldsymbol{G}\dot{\boldsymbol{q}}$ term, where $\boldsymbol{G}$ is a skew-symmetric matrix representing gyric effects
  - Finally we can add a $\boldsymbol{H}\boldsymbol{q}$ term where $\boldsymbol{H}$ is a skew-symmetric matrix representing circulatory effects (follower forces, e.g. lift and drag)
  - This is the general form for a linear system
- Note that to obtain the linear system, we need to find the kinetic and potential energies to second order

Example: Double Pendulum

- Consider a double pendulum with masses $m_1 = m_2 = m$, angles θ_1, θ_2 from vertical, and link lengths $l_1 = l_2 = l$
- $T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$
  - $v_1 = l_1\dot{\theta}_1$
  - For v_2, we need to add the velocities of the first mass and the second mass relative to the first mass, which are in general not in the same direction
  - The relative speed is $v_2' = l_2\dot{\theta}_2$, which forms a triangle with v_1
  - Using the cosine law: $v_2^2 = v_1^2 + (v_2')^2 - 2v_1v_2'\cos(\pi - (\theta_2 - \theta_1)) = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$
  - $T = \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right)$
    - We can expand $\cos(\theta_2 - \theta_1) = 1 - \frac{1}{2}(\theta_2 - \theta_1)^2$ to second order
    - However since we already have a $\dot{\theta}_1\dot{\theta}_2$ multiplying this, it will be 4th order, which we can ignore
  - Therefore $T = \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right)$
- This gives $T = \frac{1}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{M}\dot{\boldsymbol{\theta}}$ where $\boldsymbol{M} = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix} = ml^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
- $V = mgl(1 - \cos\theta_1) + mgl(1 - \cos\theta_2 + 1 - \cos\theta_1)$

- Expanding this to second order, we get $\frac{1}{2}mgl(2\theta_1^2 + \theta_2^2)$
- Therefore $V = \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{K}\boldsymbol{\theta}$ where $\boldsymbol{K} = mgl\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$

- The equation of motion is therefore $\boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{K}\boldsymbol{\theta} = \boldsymbol{0}$

Lecture 22, Nov 30, 2023

Modal Analysis

- Consider first the unforced equation of motion, $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}$
- Substituting in the test solution $\boldsymbol{q}_0 e^{\lambda t}$ gives us $(\lambda^2\boldsymbol{M} + \boldsymbol{K})\boldsymbol{q}_0 = \boldsymbol{0}$
  - Clearly, this is satisfied for the trivial solution $\boldsymbol{q}_0 = \boldsymbol{0}$, but we want some non-quiescent solution
  - For $\boldsymbol{q}_0$ to be nonzero, we need $\det(\lambda^2\boldsymbol{M} + \boldsymbol{K}) = 0$; this resembles an eigenproblem where instead of identity we have $\boldsymbol{M}$
  - There will be multiple such λ and $\boldsymbol{q}_0$
- λ_α^2 are then the eigenvalues and $\boldsymbol{q}_\alpha$ the eigenvectors; $\det(\lambda^2\boldsymbol{M} + \boldsymbol{K}) = 0$ is the characteristic equation or eigenequation
  - In general the eigenequation gives us an n-th order polynomial in λ^2
  - Consider multiplying both sides by the Hermitian of $\boldsymbol{q}_\alpha$: $\lambda_\alpha^2\boldsymbol{q}_\alpha^H\boldsymbol{M}\boldsymbol{q}_\alpha + \boldsymbol{q}_\alpha^H\boldsymbol{K}\boldsymbol{q}_\alpha = 0$
    * $\boldsymbol{M}$ is real and symmetric, so $\boldsymbol{q}_\alpha^H\boldsymbol{M}\boldsymbol{q}_\alpha$ is real; furthermore its positive-definiteness means this is also always greater than 0
    * $\boldsymbol{K}$ is real and symmetric, so $\boldsymbol{q}_\alpha^H\boldsymbol{K}\boldsymbol{q}_\alpha$ is also real
    * $\lambda_\alpha^2 = -\dfrac{\boldsymbol{q}_\alpha^H\boldsymbol{K}\boldsymbol{q}_\alpha}{\boldsymbol{q}_\alpha^H\boldsymbol{M}\boldsymbol{q}_\alpha}$ so indeed λ is real
    * Further, $\boldsymbol{K} > 0 \implies \lambda_\alpha^2 < 0$, or $\lambda_\alpha = \pm j\omega_\alpha$
    * By extension, the $\boldsymbol{q}_\alpha$ are real
- Do $\boldsymbol{q}_\alpha$ form a basis?
  - Consider $\lambda_\alpha^2\boldsymbol{q}_\beta^T\boldsymbol{M}\boldsymbol{q}_\alpha + \boldsymbol{q}_\beta^T\boldsymbol{K}\boldsymbol{q}_\alpha = 0$ and $\lambda_\beta^2\boldsymbol{q}_\alpha^T\boldsymbol{M}\boldsymbol{q}_\beta + \boldsymbol{q}_\alpha^T\boldsymbol{K}\boldsymbol{q}_\beta = 0$
  - Subtracting the two equations gives $(\lambda_\alpha^2 - \lambda_\beta^2)\boldsymbol{q}_\alpha^T\boldsymbol{M}\boldsymbol{q}_\beta = 0$ (note we can do this since the terms are scalars)
  - Then $\boldsymbol{q}_\alpha^T\boldsymbol{M}\boldsymbol{q}_\beta = \begin{cases} > 0 & \lambda_\alpha^2 = \lambda_\beta^2 \\ 0 & \lambda_\alpha^2 \neq \lambda_\beta^2\end{cases}$
  - WLOG normalize the $\boldsymbol{q}$ vectors with respect to $\boldsymbol{M}$, then $\boldsymbol{q}_\alpha^T\boldsymbol{M}\boldsymbol{q}_\beta = \delta_{\alpha\beta} \implies \boldsymbol{Q}^T\boldsymbol{M}\boldsymbol{Q} = \boldsymbol{1}$ where $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{q}_1 & \cdots & \boldsymbol{q}_m\end{bmatrix}$
    * Note we might get repeated eigenvalues, but we can always diagonalize due to the symmetry of $\boldsymbol{M}$
  - Plugging back into the first equation, $\lambda_\alpha^2\delta_{\alpha\beta} + \boldsymbol{q}_\alpha^T\boldsymbol{K}\boldsymbol{q}_\beta^T \implies \boldsymbol{q}_\alpha^T\boldsymbol{K}\boldsymbol{q}_\beta^T = \begin{cases} -\lambda_\alpha^2 & \alpha = \beta \\ 0 & \alpha \neq \beta\end{cases}$
  - So we can also write $\boldsymbol{Q}^T\boldsymbol{K}\boldsymbol{Q} = -\Lambda^2$
- Let $\boldsymbol{q}(t) = \sum_{\beta=1}^{n}\boldsymbol{q}_\beta\eta_\beta(t)$
  - $$\begin{aligned}&\sum_{\beta=1}^{n}\boldsymbol{M}\boldsymbol{q}_\beta\ddot{\eta}_\beta + \sum_{\beta=1}^{n}\boldsymbol{K}\boldsymbol{q}_\beta\eta_\beta = \boldsymbol{f}\\ \implies &\sum_{\beta=1}^{n}\boldsymbol{q}_\alpha^T\boldsymbol{M}\boldsymbol{q}_\beta\ddot{\eta}_\beta + \sum_{\beta=1}^{n}\boldsymbol{q}_\alpha^T\boldsymbol{K}\boldsymbol{q}_\beta\eta_\beta = \boldsymbol{q}_\alpha^T\boldsymbol{f}\\ \implies &\sum_{\beta=1}^{n}\delta_{\alpha\beta}\ddot{\eta}_\beta + \sum_{\beta=1}^{n}-\lambda_\alpha^2\eta_\beta = \boldsymbol{q}_\alpha^T\boldsymbol{f}\\ \implies &\ddot{\eta}_\alpha - \lambda_\alpha^2\eta_\alpha = f_\alpha\end{aligned}$$

  - We have uncoupled the system of differential equations
  - $\boldsymbol{q}_\beta$ are the *mode shapes*, and $\eta_\beta(t)$ are the *modal coordinates*; $\boldsymbol{q}_\alpha \eta_\alpha$ is a *mode* of vibration
  - Note that if $\boldsymbol{K}$ were positive definite, we would get all negative λ_α^2, giving oscillatory motion; then $\omega_\alpha^2 = -\lambda_\alpha^2$ are the *natural frequencies* of vibration

Double Pendulum Revisited

- Recall: $\boldsymbol{M} = ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \boldsymbol{K} = mgl \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
- Clearly $\boldsymbol{K}$ is positive definite here, so let's write λ^2 as $-\omega^2$
- We want to solve $\det(-\omega_\alpha^2 \boldsymbol{M} + \boldsymbol{K}) = 0$
- Let $\mu_\alpha^2 = -\omega_\alpha^2 \dfrac{l}{g}$, so that $\det \begin{bmatrix} -2\mu_\alpha^2 + 2 & -\mu_\alpha^2 \\ -\mu_\alpha^2 & -\mu_\alpha^2 + 1 \end{bmatrix} = 0$
- Expanded: $\mu^3 - 4\mu^2 + 2 = 0 \implies \mu^2 = 2 \pm \sqrt{2}$
- Therefore the modal frequencies are $\omega_1 = \sqrt{(2 - \sqrt{2})\dfrac{g}{l}}, \omega_2 = \sqrt{(2 + \sqrt{2})\dfrac{g}{l}}$
- Solve the eigenequation to get $\dfrac{\theta_{1,2}}{\theta_{1,1}} = \sqrt{2}$ and $\dfrac{\theta_{2,2}}{\theta_{2,1}} = -\sqrt{2}$
- For each of the modes, at any time, the ratio of the coordinates remains the same
- Notice that in the second mode, we have a *node* – a point that does not move
  - In general, for an n degree of freedom system, we will have n modes; mode n will have $n-1$ nodes

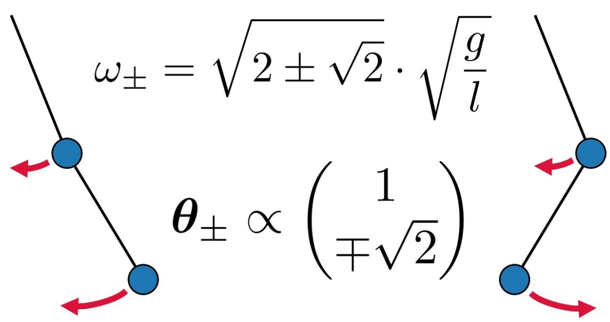

Figure 12: Vibrational modes of the double pendulum.

Lecture 23, Dec 5, 2023

Exam Review

- Consider a system with a mass m on a horizontal frictionless plane, connected to inertial space by a spring of stiffness $k = \dfrac{mg}{l}$, with x measured from the equilibrium position; a pendulum of length l and mass m is connected to the mass, with θ measured from vertical
  a. Derive the potential energy for the system
    - $V = \dfrac{1}{2}kx^2 - mgl\cos\theta$
    - Approximate to second order: $V = \dfrac{1}{2}kx^2 - mgl\left(1 - \dfrac{1}{2}\theta^2\right) = \dfrac{1}{2}kx^2 - mgl + \dfrac{1}{2}mgl\theta^2$
    - We want this in the form of $\dfrac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}$, ignoring constant terms
    - Let us define $\boldsymbol{q} = \begin{bmatrix} x \\ l\theta \end{bmatrix}$ so that the units are consistent

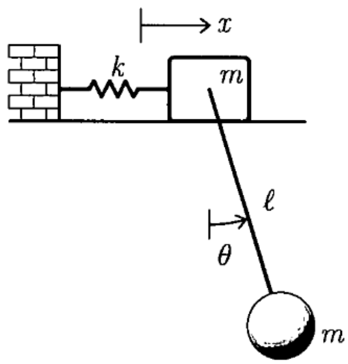

Figure 13: Example problem 1.

   - $\boldsymbol{K} = \begin{bmatrix} k & 0 \\ 0 & \dfrac{mg}{l} \end{bmatrix} = k\mathbf{1}$ (note no 1/2 in the matrix, since the factor is outside)
b. Derive the kinetic energy for the system
   - The velocity of the bob is the vector sum of the block's velocity and the bob's velocity relative to the block
   - By the cosine law: $v^2 = \dot{x}^2 + l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos(\pi - \theta) = \dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta$
   - $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mv^2 = \frac{1}{2}m\left(2\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right)$
   - We want this in the form of $\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$
   - To second order: $T = \frac{1}{2}m(2\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta})$
   - $\boldsymbol{M} = m\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (note no l terms in the matrix since these are in $\boldsymbol{q}$ itself)
c. What are the linearized equations of motion?
   - $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \mathbf{0}$
   - Notice that $\boldsymbol{K}$ is symmetric and positive definite, so all modes are stable (purely imaginary eigenvalues)
d. Determine the frequencies of vibration
   - This requires us to solve for the eigenvalues
   - $\det(\lambda^2\boldsymbol{M} + \boldsymbol{K}) = 0$
   - Since we know λ are purely imaginary, let $\lambda^2 = -\omega^2$
   - $\det(-\omega^2\boldsymbol{M} + \boldsymbol{K}) = \det\left(\begin{bmatrix} k - 2m\omega^2 & -\omega^2 m \\ -\omega^2 m & k - m\omega^2 \end{bmatrix}\right) = 0$
   - Let $\mu^2 = \dfrac{\omega^2 m}{k}$, then $\det\left(\begin{bmatrix} -2\mu^2 + 1 & -\mu^2 \\ -\mu^2 & -\mu^2 + 1 \end{bmatrix}\right) = 0$ (after dividing through by k)
   - $\mu^4 - 2\mu^2 + 1 = 0 \implies \mu^2 = \dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$
   - $\omega^2 = \dfrac{k}{m}\left(\dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}\right) \implies \omega_1 = \sqrt{\dfrac{k}{m}\left(\dfrac{3}{2} - \dfrac{\sqrt{5}}{2}\right)}, \omega_2 = \sqrt{\dfrac{k}{m}\left(\dfrac{3}{2} + \dfrac{\sqrt{5}}{2}\right)}$
e. Determine and sketch the mode shapes of vibration
   - This requires us to solve for the eigenvectors
   - Plug in ω^2 to $(-\omega_\alpha^2\boldsymbol{M} + \boldsymbol{K})\boldsymbol{q}_\alpha = \mathbf{0}$
   - $\boldsymbol{q}_1 \propto \begin{bmatrix} -1 + \sqrt{5} \\ 3 - \sqrt{5} \end{bmatrix}, \boldsymbol{q}_2 \propto \begin{bmatrix} -1 - \sqrt{5} \\ 3 + \sqrt{5} \end{bmatrix}$
   - The first node has both the block and pendulum on the same side, while the second node has the block and pendulum on different sides in opposing motion

- In general it usually helps to make the coordinates dimensionally consistent, so that the mass and stiffness matrices are dimensionally consistent, which usually makes the math easier

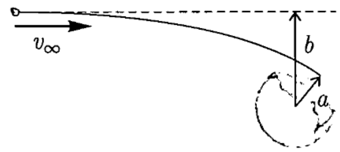

Figure 14: Example problem 2.

- Consider a meteor approaching from infinity with $v_\infty = \sqrt{\frac{\mu}{a}}$, where a is the radius of the Earth and μ is the reduced mass of the meteor and Earth; let the perpendicular distance from the centre of the Earth to the asymptotes of the hyperbolic orbit be b; what would b be if the meteor were to just skim the surface of the Earth?
  - We know this orbit will be hyperbolic, since if it were parabolic, we'd have $v_\infty = 0$
  - The specific energy is $e = \frac{1}{2}v^2 - \frac{\mu}{r}$
  - The specific angular momentum at infinity is $h = bv_\infty$ (since b is the moment arm, and v_∞ is the velocity)
  - When skimming the Earth, we have $h = av_p$, but due to conservation of angular momentum we have $av_p = bv_\infty$ so $b = \frac{av_p}{v_\infty}$
  - To get v_p we use energy conservation: at infinity $e = \frac{1}{2}v_\infty^2$ (since $r \to \infty$); when skimming the Earth, $e = \frac{1}{2}v_p^2 - \frac{\mu}{a}$
  - Therefore $v_p^2 = 2\left(\frac{1}{2}v_\infty^2 + \frac{\mu}{a}\right) = \frac{3\mu}{a} \implies v_p = \sqrt{3}v_\infty$
  - Therefore $b = \frac{a\sqrt{3}v_\infty}{v_\infty} = \sqrt{3}a$

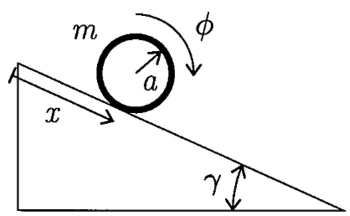

Figure 15: Example problem 3.

- Consider a uniform hoop of mass m and radius a rolling without slipping on an incline of angle γ; the distance travelled by the hoop is x and its rotation angle is ϕ
  a. What is the constraint in Pffafian form?
     - $\mathrm{d}x - a\,\mathrm{d}\phi = 0$
  b. Derive the equations of motion using Lagrange multipliers and solve for the translational acceleration down the incline
     - $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}m(\dot{x}^2 + a^2\dot{\phi}^2)$
     - For the hoop, $I = ma^2$ since all the mass is concentrated at a radius of a
     - $V = mgh = -mgx\sin\gamma$
     - No virtual work since the constraint forces do no work
     - In the form $\Xi_1\,\mathrm{d}x + \Xi_2\,\mathrm{d}\phi = 0$ we have $\Xi_1 = 1, \Xi_2 = -a$
     - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}, \frac{\partial L}{\partial x} = mg\sin\gamma, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ma^2\ddot{\phi}, \frac{\partial L}{\partial \phi} = 0$

- Recall: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{q}_k}\right) - \dfrac{\partial L}{\partial q_k} = \sum_j \lambda_j \Xi_{jk}$ so the equations of motion are:
  - $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = \lambda \Xi_1 \implies m\ddot{x} - mg\sin\gamma = \lambda$
  - $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = \lambda \Xi_2 \implies ma^2\ddot{\phi} = -a\lambda$
  - $\Xi_1 \,\mathrm{d}x + \Xi_2 \,\mathrm{d}\phi = 0 \implies \dot{x} - a\dot{\phi} = 0$ which we can integrate
- Solving gives $\ddot{x} = \dfrac{1}{2} g \sin\gamma$