Lecture 9, Feb 1, 2022

Basis

- Fundamental Theorem of Linear Algebra: Let $\mathcal{V}$ be a vector space spanned by n vectors. If a set of m vectors from $\mathcal{V}$ is linearly independent, then $m \leq n$.
  - This is equivalent to saying if $m > n$ then any set of m vectors from $\mathcal{V}$ is linearly dependent (this is the contrapositive statement: if $A \implies B$, then $\neg B \implies \neg A$)
  - Proof by contraposition: Let $m > n$, we show that this implies a set of m vectors is dependent.
    * Consider a set of m vectors $\{ \boldsymbol{u}_1, \cdots, \boldsymbol{u}_m \}$ and let $\operatorname{span}\{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \} = \mathcal{V}$
    * Since $\boldsymbol{u}_j \in \mathcal{V}$, $\boldsymbol{u}_j = \sum_{i=1}^{n} a_{ij}\boldsymbol{v}_i$ and so $\sum_{j=1}^{m} x_j \boldsymbol{u}_j = \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} a_{ij}\boldsymbol{v}_i \right)$
      $$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right) \boldsymbol{v}_i$$
    * Set $\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right) \boldsymbol{v}_i = \mathbf{0}$; this will be satisfied if each $\sum_{j=1}^{m} a_{ij} x_j = 0$; this is a set of linear equations $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ where $\boldsymbol{A}$ is $n \times m$; since we have $m > n$ there are infinite number of solutions to this system, i.e. there exist a nontrivial solution, therefore not all x_j have to be 0, so the set $\{ \boldsymbol{u}_1, \cdots, \boldsymbol{u}_m \}$ is linearly dependent
- Define the *basis* for a vectors space $\mathcal{V}$ to be a set of vectors $\{ \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n \}$ that are linearly independent and $\operatorname{span}\{ \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m \} = \mathcal{V}$
  - Every basis for a given vector space contains the same number of vectors:
    * Let $E = \{ \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n \}$ and $F = \{ \boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_n \}$ be bases
    * Consider E to be linearly independent and F to span $\mathcal{V}$, then by the fundamental theorem $n \leq m$; consider F to be linearly independent and E to span $\mathcal{V}$, then by the fundamental theorem $m \leq n$, therefore $m = n$
  - We say that a basis *generates* $\mathcal{V}$
- Definition: The *dimension* of a vector space $\mathcal{V}$, denoted $\dim \mathcal{V}$, is the number of vectors in any of its bases
  - Note: Define $\dim \{ \mathbf{0} \} = 0$