Lecture 8, Jan 31, 2022

Equivalence of Spans

- We can show equivalence of sets $U = V$ by showing $U \subseteq V$ and $V \subseteq U$
- To show equivalence of spans, we do the same and show both spans are subsets of the other
  - It is sufficient to show that each member of the spanning set is in the other span; if $\boldsymbol{u}_i = \sum_{k=1}^{n} \boldsymbol{\alpha}_{ik} v_k$, then $\boldsymbol{u} = \sum_{i=1}^{m} \lambda_i \boldsymbol{u}_i = \sum_{i=1}^{m} \lambda_i \left(\sum_{k=1}^{n} \alpha_{ik} \boldsymbol{v}_k \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{m} \right) \boldsymbol{v}_k = \sum_{k=1}^{n} \mu_k \boldsymbol{v}_k \implies \boldsymbol{u} \in \operatorname{span}\{ \boldsymbol{v}_j \}$
- Example: $\operatorname{span}\{ \boldsymbol{u}, \boldsymbol{v} \} = \operatorname{span}\{ \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v} \}$?
  - $\{ \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v} \} \subseteq \operatorname{span}\{ \boldsymbol{u}, \boldsymbol{v} \}$ since they're both linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$
  - $\{ \boldsymbol{u}, \boldsymbol{v} \} \subseteq \operatorname{span}\{ \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v} \}$ since $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}) + \frac{1}{2}(\boldsymbol{u}-\boldsymbol{v})$ and $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}) - \frac{1}{2}(\boldsymbol{u}-\boldsymbol{v})$
- Proposition II: Let $\mathcal{U} = \operatorname{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \sqsubseteq \mathcal{V}$. If $\mathcal{W}$ is a subspace of $\mathcal{V}$ containing the vectors $\operatorname{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ then $\mathcal{U} \sqsubseteq \mathcal{W}$.
  - Any vector in $\mathcal{U}$ is a linear combination of those vectors, and since those vectors are in $\mathcal{W}$, $\mathcal{W}$ contains all linear combinations of those vectors

Linear Independence

- Linear independence: A set of vectors $\operatorname{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ is *linearly independent* if and only if $\sum_{j=1}^{n} \lambda_j \boldsymbol{v}_j = \boldsymbol{0} \implies \lambda_j = 0$
  - Example: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ are independent: $\lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \lambda_1 = \lambda_2 = 0$
  - Example: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ are not independent: $\lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ can be satisfied with $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = -1 \end{cases}$
- Proposition I: If $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \subset \mathcal{V}$ is linearly independent and $\boldsymbol{v} = \sum_{j=1}^{n} \lambda_j \boldsymbol{v}_j$ for all $\boldsymbol{v} \in \mathcal{V}$ then λ_j are uniquely determined, i.e. there is only one way to construct any vector
  - Proof: Assume that λ_j are not uniquely determined. Let $\boldsymbol{v} = \sum_{j=1}^{n} \lambda_j \boldsymbol{v}_j = \sum_{j=1}^{n} \mu_j \boldsymbol{v}_j$ then $\boldsymbol{0} = \boldsymbol{v} - \boldsymbol{v} = \sum_{j=1}^{n} (\lambda_j - \mu_j) = \boldsymbol{v}_j$ and because the set is linearly independent $\lambda_j - \mu_j = 0 \implies \lambda_j = \mu_j$, so λ_j are uniquely determined
- This generalizes to any kind of vector, e.g. functions
  - e.g. to show $\{ \sin x, \cos x \}$ are linearly independent we show $\lambda_1 f + \lambda_2 g = z \implies \lambda_1 = \lambda_2 = 0$ where $z : \mathbb{R} \mapsto \{ 0 \}$
    - We want to show $\lambda_1 \cos x + \lambda_2 \sin x = 0 \forall x \in \mathbb{R}$; we can consider $x = 0 \implies \lambda 1 = 0$, and $x = \frac{\pi}{2} \implies \lambda_2 = 0$