Lecture 7, Jan 28, 2022

Linear Combination and Span

- Definition: A vector $\boldsymbol{v} \in \mathcal{V}$ is a linear combination of $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \subset \mathcal{V}$ if and only if it can be written as $\boldsymbol{v} = \sum_{j=1}^{n} \lambda_j \boldsymbol{v}_j$ for $\lambda_j \in \Gamma$
  - Note the use of $\subset$ instead of $\subseteq$ because for now we want to keep the set finite
- Definition: The *span* of $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \subset \mathcal{V}$ is denoted: $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} = \left\{ \boldsymbol{v} \;\middle|\; \boldsymbol{v} = \sum_{j=1}^{n} \lambda_j \boldsymbol{v}_j, \forall \lambda_j \in \Gamma \right\}$, i.e. all the vectors that can be written as a linear combination of this set of vectors
  - Example: ${}^3\mathbb{R} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
  - $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ is the *spanning set* of vectors (for now, this set will be finite, but the span itself is infinite)
- Proposition I: The span of $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \in \mathcal{V}$ is a subspace of $\mathcal{V}$
  - Proof:
    * $\mathcal{SI}$: $\mathbf{0} = \sum_{j=1}^{n} 0\boldsymbol{v}_j$ therefore $\mathbf{0}$ in this subset
    * $\mathcal{SII}$: Let $\boldsymbol{u} \in \mathcal{V} = \sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j$ and $\boldsymbol{w} \in \mathcal{V} = \sum_{j=1}^{n} \beta_j \boldsymbol{v}_j$ then $\boldsymbol{u} + \boldsymbol{w} = \sum_{j=1}^{n} (\alpha_j + \beta_j) \boldsymbol{v}_j$
    * $\mathcal{SIII}$: $\boldsymbol{u} \in \mathcal{V} = \sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j$, then $\lambda \boldsymbol{u} = \lambda \sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j = \sum_{j=1}^{n} (\lambda \alpha_j) \boldsymbol{v}_j$