Lecture 6, Jan 25, 2022

Subspaces

- A subset $\mathcal{U}$ of $\mathcal{V}$ is a subspace of $\mathcal{V}$ iff $\mathcal{U}$ is itself a vector space over the same field Γ with the same vector addition and scalar multiplication operations of $\mathcal{V}$
  - $X \subseteq Y \iff \forall x \in X \implies x \in Y$
  - In this case the subset is not strict, i.e. $\mathcal{U} = \mathcal{V}$ is allowed
  - Every $\mathcal{V}$ has two subspaces, the space itself, and the subspace of only zero: $\mathcal{U} = V$ and $\mathcal{U} = \{ \mathbf{0} \}$
  - Sometimes the notation $\mathcal{U} \sqsubseteq \mathcal{V}$ is used
- Theorem 1: Subspace test: $\mathcal{U} \sqsubseteq \mathcal{V}$ iff for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}$ and all $\alpha \in \Gamma$:
  1. Zero: $\exists \mathbf{0} \in \mathcal{U} \ni \boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
  2. Closure under addition: $\boldsymbol{u} + \boldsymbol{v} \in \mathcal{U}$
  3. Closure under scalar multiplication: $\alpha\boldsymbol{u} \in \mathcal{U}$
- Proof of the subspace test:
  - $\mathcal{U} \sqsubseteq \mathcal{V} \implies (\mathcal{SI}, \mathcal{SII}, \mathcal{SIII})$: By definition $\mathcal{U}$ is a vector space, therefore it automatically satisfies all 3 axioms
  - $(\mathcal{SI}, \mathcal{SII}, \mathcal{SIII}) \implies \mathcal{U} \sqsubseteq \mathcal{V}$:
    * $\mathcal{AI}$: Implied by $\mathcal{SII}$
    * $\mathcal{AII}$: Automatically true since $\boldsymbol{u} \in \mathcal{U} \implies \boldsymbol{u} \in \mathcal{V}$ and the addition operator is associative in $\mathcal{V}$ (i.e. inherited from $\mathcal{V}$)
    * $\mathcal{AIII}$: Implied by $\mathcal{SI}$
    * $\mathcal{AIV}$: We have proven previously that $(-1)\boldsymbol{u}$ is the additive inverse of $\boldsymbol{u}$; we also know $(-1)\boldsymbol{u} \in \mathcal{U}$ by $\mathcal{SIII}$, so an inverse exists
    * $\mathcal{MI}$: Implied by $\mathcal{SIII}$
    * $\mathcal{MII} - \mathcal{MIII}$: Inherited from $\mathcal{V}$
    * $\mathcal{MIV}$: $1\boldsymbol{u} \in \mathcal{U}$ by $\mathcal{SIII}$ and $\boldsymbol{u} \in \mathcal{V}$ so $1\boldsymbol{u} = \boldsymbol{u} \in \mathcal{U}$
- Example: $\operatorname{im} A = \{ \boldsymbol{y} \mid \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \forall \boldsymbol{x} \in {}^n\mathbb{R} \} \subseteq {}^m\mathbb{R}$ for $\boldsymbol{A} \in {}^m\mathbb{R}^n$
  - Since ${}^m\mathbb{R}$ is a vector space over $\mathbb{R}$ we only need to do the subspace test
  - $\mathcal{SI}$: Satisfied since $\mathbf{0} = \boldsymbol{A}\mathbf{0} \implies \mathbf{0} \in \operatorname{im} \boldsymbol{A}$
  - $\mathcal{SII}$: $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \operatorname{im} \boldsymbol{A} \implies \boldsymbol{y}_1 = \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{y}_2 = \boldsymbol{A}\boldsymbol{x}_2 \implies \boldsymbol{y}_1 + \boldsymbol{y}_2 = \boldsymbol{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$
  - $\mathcal{SIII}$: $\boldsymbol{y} \in \operatorname{im} \boldsymbol{A} \implies \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \implies \alpha\boldsymbol{y} = \alpha(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\alpha\boldsymbol{x}) \in \operatorname{im} \boldsymbol{A}$
- Example: $\ker \boldsymbol{A} = \{ \boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \mathbf{0} \}$ for $\boldsymbol{A} \in {}^m\mathbb{R}^n$ (kernel or null space of A)
  - $x \in {}^n\mathbb{R}$ so we can apply the subspace test
  - $\mathcal{SI}$: $\mathbf{0} \in \ker \boldsymbol{A}$ because $\boldsymbol{A}\mathbf{0} = \mathbf{0}$
  - $\mathcal{SI}$: $\boldsymbol{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \boldsymbol{x}_1 + \boldsymbol{x}_2 \in \ker \boldsymbol{A}$
  - $\mathcal{SIII}$: $\boldsymbol{A}(\alpha\boldsymbol{x}) = \alpha(\boldsymbol{A}\boldsymbol{x}) = \alpha\mathbf{0} = \mathbf{0} \implies \alpha\boldsymbol{x} \in \ker \boldsymbol{A}$