Lecture 5, Jan 24, 2022

More Vector Space Properties

- Proposition V: Properties of zero: For all $\boldsymbol{v} \in \mathcal{V}$ and all $\alpha \in \Gamma$:
  1. $0\boldsymbol{v} = \boldsymbol{0}$
     - $0\boldsymbol{v} = 0\boldsymbol{v} + \boldsymbol{0}$ by $\mathcal{AIII}$
     - $0\boldsymbol{v} = (0+0)\boldsymbol{v} = 0\boldsymbol{v} + 0\boldsymbol{v}$ by $\mathcal{MIII}(a)$ and scalar addition properties
     - By the transitive property $0\boldsymbol{v} + \boldsymbol{0} = 0\boldsymbol{v} + 0\boldsymbol{v}$, then by the cancellation theorem $0\boldsymbol{v} = \boldsymbol{0}$
  2. $\alpha\boldsymbol{0} = \boldsymbol{0}$
     - $\alpha\boldsymbol{0} = \alpha(\boldsymbol{0} + \boldsymbol{0}) = \alpha\boldsymbol{0} + \alpha\boldsymbol{0}$
     - Rest of the proof follows like above
  3. If $\alpha\boldsymbol{v} = \boldsymbol{0}$ then either $\alpha = 0$ or $\boldsymbol{v} = \boldsymbol{0}$
     - Either $\alpha = 0$ or $\alpha \neq 0$; if $\alpha = 0$ then $0\boldsymbol{v} = \boldsymbol{0}$ follows by 1, so we only need to consider $\alpha \neq 0$
     - $$\begin{aligned} \boldsymbol{v} &= 1\boldsymbol{v} && \mathcal{MIV} \\ &= (\alpha^{-1}\alpha)\boldsymbol{v} && \text{Properties of scalars} \\ &= \alpha^{-1}(\alpha\boldsymbol{v}) && \mathcal{MII} \\ &= \alpha^{-1}\boldsymbol{0} && \text{Given} \\ &= \boldsymbol{0} && \text{Prop. V.2} \end{aligned}$$
     - Therefore either $\alpha = 0$, or if $\alpha \neq 0$, then $\boldsymbol{v} = \boldsymbol{0}$
- Proposition VI: For all $\boldsymbol{v} \in \mathcal{V}$ and $\alpha \in \Gamma$, $(-\alpha)\boldsymbol{v} = -(\alpha\boldsymbol{v}) = \alpha(-\boldsymbol{v})$
  - $$\begin{aligned} \alpha\boldsymbol{v} + (-\alpha\boldsymbol{v}) &= (\alpha - \alpha)\boldsymbol{v} && \mathcal{MIII}(a) \\ &= 0\boldsymbol{v} && \text{Properties of scalars} \\ &= \boldsymbol{0} && \text{Prop. V.1} \end{aligned}$$
  - Since $\alpha\boldsymbol{v} + (-(\alpha\boldsymbol{v})) = \boldsymbol{0}$ by $\mathcal{AIII}$, by the transitive property and cancellation $-(\alpha\boldsymbol{v}) = (-\alpha)\boldsymbol{v}$
  - $$\begin{aligned} \alpha\boldsymbol{v} + \alpha(-\boldsymbol{v}) &= \alpha(\boldsymbol{v} - \boldsymbol{v}) && \mathcal{MIII}(b) \\ &= \alpha\boldsymbol{0} && \mathcal{AIV} \\ &= \boldsymbol{0} && \text{Prop. V.2} \end{aligned}$$
  - It follows then that $\alpha(-\boldsymbol{v}) = -(\alpha\boldsymbol{v}) = (-\alpha\boldsymbol{v})$
  - Consider $\alpha = 1$, then $-(1\boldsymbol{v}) = -\boldsymbol{v} = (-1)\boldsymbol{v}$, so the additive inverse is always -1 times the vector!