Lecture 4, Jan 18, 2022

Commutativity and Other Properties of Vector Spaces

- What about the associative property $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$?
  - This can be proven from the other properties, but first we will start with some other axioms
- There are some axioms that are one sided such as $\mathcal{AIII}$ and $\mathcal{AIV}$ (additive identity and inverse); we will prove that these are two sided under the other axioms
- Proposition I: For every $\boldsymbol{u}, -\boldsymbol{u} \in \mathcal{V}, -\boldsymbol{u}+\boldsymbol{u}=\mathbf{0}$ (i.e. property $\mathcal{AIV}$ but commutative)
  - Proof: $$\begin{aligned} -\boldsymbol{u}+\boldsymbol{u} &= (-\boldsymbol{u}+\boldsymbol{u})+\mathbf{0} && \mathcal{AIII} \\ &= (-\boldsymbol{u}+\boldsymbol{u})+(-\boldsymbol{u}+(-(-\boldsymbol{u}))) && \mathcal{AIV} \\ &= -\boldsymbol{u}+(\boldsymbol{u}+(-\boldsymbol{u}))+(-(-\boldsymbol{u})) && \mathcal{AII} \\ &= -\boldsymbol{u}+\mathbf{0}+(-(-\boldsymbol{u})) && \mathcal{AIV} \\ &= -\boldsymbol{u}+(-(-\boldsymbol{u})) && \mathcal{AIII} \\ &= \mathbf{0} && \mathcal{AIV} \end{aligned}$$
  - Thus $\mathcal{AIV}$ is commutative, and we can say that the additive inverse of $-\boldsymbol{u}$ is just $\boldsymbol{u}$
- Proposition II: For every $u \in \mathcal{V}, \mathbf{0}+\boldsymbol{u}=\boldsymbol{u}$ (i.e. property $\mathcal{AIII}$ but commutative)
  - Proof: $$\begin{aligned} \mathbf{0}+\boldsymbol{u} &= (\boldsymbol{u}+(-\boldsymbol{u}))+\boldsymbol{u} && \mathcal{AIV} \\ &= \boldsymbol{u}+(-\boldsymbol{u}+\boldsymbol{u}) && \mathcal{AII} \\ &= \boldsymbol{u}+\mathbf{0} && \text{Prop. I} \\ &= \boldsymbol{u} && \mathcal{AIII} \end{aligned}$$
- Theorem I: Cancellation theorem: If $\boldsymbol{u}+\boldsymbol{w}=\boldsymbol{v}+\boldsymbol{w}$ then $\boldsymbol{u}+\boldsymbol{v}$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ (this also applies for $\boldsymbol{w}+\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{v}$)
  - Proof: $$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}+\mathbf{0} && \mathcal{AIII} \\ &= \boldsymbol{u}+(\boldsymbol{w}+-\boldsymbol{w}) && \mathcal{AIV} \\ &= (\boldsymbol{u}+\boldsymbol{w})+(-\boldsymbol{w}) && \mathcal{AII} \\ &= (\boldsymbol{v}+\boldsymbol{w})+(-\boldsymbol{w}) && \text{given} \\ &= \boldsymbol{v}+(\boldsymbol{w}+(-\boldsymbol{w})) && \mathcal{AII} \\ &= \boldsymbol{v}+\mathbf{0} && \mathcal{AIV} \\ &= \boldsymbol{v} && \mathcal{AIII} \end{aligned}$$
- Note: a *theorem* and *proposition* are basically the same thing here, but typically theorem is used for bigger results
- Define subtraction: $\boldsymbol{u}-\boldsymbol{v}=\boldsymbol{u}+(-\boldsymbol{v})$
- Proposition III:
  1. The zero $\mathbf{0} \in \mathcal{V}$ is unique
     - Proof: Let $\mathbf{0}'$ be another zero, then $\boldsymbol{u}+\mathbf{0}'=\boldsymbol{u}=\boldsymbol{u}+\mathbf{0}$ so by the cancellation theorem, $\mathbf{0}'=\mathbf{0}$
  2. The inverse is unique
  3. $-(-\boldsymbol{u})=\boldsymbol{u}$
- Proposition IV: For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}, \boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$

– Proof:
$$\begin{aligned}
\boldsymbol{u}+\boldsymbol{v} &= \mathbf{0}+(\boldsymbol{u}+\boldsymbol{v})+\mathbf{0} && \text{Prop. II and } \mathcal{AIII} \\
&= (-\boldsymbol{v}+\boldsymbol{v})+(\boldsymbol{u}+\boldsymbol{v})+(\boldsymbol{u}+(-\boldsymbol{u})) && \text{Prop. I and } \mathcal{AIV} \\
&= -\boldsymbol{v}+((\boldsymbol{v}+\boldsymbol{u})+(\boldsymbol{v}+\boldsymbol{u}))+(-\boldsymbol{u}) && \mathcal{AII} \\
&= -\boldsymbol{v}+(1(\boldsymbol{v}+\boldsymbol{u})+1(\boldsymbol{v}+\boldsymbol{u}))+(-\boldsymbol{u}) && \mathcal{MIV} \\
&= -\boldsymbol{v}+(1+1)(\boldsymbol{v}+\boldsymbol{u})+(-\boldsymbol{u}) && \mathcal{MIII} \\
&= -\boldsymbol{v}+((1+1)\boldsymbol{v}+(1+1)\boldsymbol{u})+(-\boldsymbol{u}) && \mathcal{MIII} \\
&= -\boldsymbol{v}+(1\boldsymbol{v}+1\boldsymbol{v}+1\boldsymbol{u}+1\boldsymbol{u})+(-\boldsymbol{u}) && \mathcal{MIII} \\
&= -\boldsymbol{v}+(\boldsymbol{v}+\boldsymbol{v}+\boldsymbol{u}+\boldsymbol{u})+(-\boldsymbol{u}) && \mathcal{MIV} \\
&= (-\boldsymbol{v}+\boldsymbol{v})+\boldsymbol{v}+\boldsymbol{u}+(\boldsymbol{u}+(-\boldsymbol{u})) && \mathcal{AII} \\
&= \mathbf{0}+\boldsymbol{v}+\boldsymbol{u}+\mathbf{0} && \text{Prop. I and } \mathcal{AIV} \\
&= \boldsymbol{v}+\boldsymbol{u} && \text{Prop. II and } \mathcal{AIII}
\end{aligned}$$