Lecture 34, Apr 8, 2022

The Matrix Exponential

- Consider the scalar case where $\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- Let $\boldsymbol{X} \in {}^n\mathbb{R}^n$, then define $e^{\boldsymbol{X}} = \exp \boldsymbol{X} = \sum_{k=0}^{\infty} \frac{\boldsymbol{X}^k}{k!}$
  - Note we have to define $\boldsymbol{X}^0 = \mathbf{1}$
- We know that in the scalar case $\dot{x} = ax$ has solution $x(t) = x_0 e^{at}$; can we do the same for the matrix exponential?
- $e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} t^k \implies \frac{\mathrm{d}}{\mathrm{d}t} e^{\boldsymbol{A}t} = \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^k}{(k-1)!} t^{k-1} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^{k+1}}{k!} t^k = \boldsymbol{A} \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} t^k = \boldsymbol{A} e^{\boldsymbol{A}t}$
- Therefore in the general case of $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$ the solution is $e^{\boldsymbol{A}t}\boldsymbol{x}_0$
- How do we actually compute $e^{\boldsymbol{A}t}$?
  - If $\boldsymbol{A}$ is diagonalizable, then $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$, so $\boldsymbol{A}^n = (\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1})(\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}) \cdots (\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}) = \boldsymbol{P}\boldsymbol{\Lambda}^n\boldsymbol{P}^{-1}$
  - $\boldsymbol{\Lambda}^n$ is easy to compute, since $\boldsymbol{\Lambda}$ is diagonal, we simply take the diaongal entries to the nth power
- Using this result,

$$
\begin{aligned}
e^{\boldsymbol{A}t} &= \sum_{k=0}^{\infty} \boldsymbol{P} \frac{\boldsymbol{\Lambda}^n}{k!} \boldsymbol{P}^{-1} t^k \\
&= \boldsymbol{P} \left(\sum_{k=0}^{\infty} \frac{\boldsymbol{\Lambda}^k}{k!} t^k \right) \boldsymbol{P}^{-1} \\
&= \boldsymbol{P} e^{\boldsymbol{\Lambda}t} \boldsymbol{P}^{-1} \\
&= \boldsymbol{P} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} t^k & 0 & \cdots \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} t^k & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \boldsymbol{P}^{-1} \\
&= \boldsymbol{P} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots \\ 0 & e^{\lambda_2 t} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \boldsymbol{P}^{-1}
\end{aligned}
$$