Lecture 33, Apr 5, 2022

Solving Differential Equations with Diagonalization

- Given $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$, how do we get $\boldsymbol{x}(t)$?
- Assume $\boldsymbol{A}$ is diagonalizable, then $\dot{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}\boldsymbol{x}$
- We can use $\boldsymbol{P}$ as a transition matrix; set $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{\eta}$, then
$$\begin{aligned} &\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \\ \implies &\boldsymbol{P}\dot{\boldsymbol{\eta}} = (\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1})(\boldsymbol{P}\boldsymbol{\eta}) \\ \implies &\boldsymbol{P}\dot{\boldsymbol{\eta}} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{\eta} \\ \implies &\dot{\boldsymbol{\eta}} = \boldsymbol{\Lambda}\boldsymbol{\eta} \end{aligned}$$
  - Since $\boldsymbol{\Lambda}$ is diagonal, we have now decoupled the system!
  - Each equation becomes $\dot{\boldsymbol{\eta}}_\alpha = \lambda_\alpha \boldsymbol{\eta}_\alpha$, so each solution is $\boldsymbol{\eta}_\alpha(t) = c_\alpha e^{\lambda_\alpha t}$
- The full solution becomes $\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{\eta}(t) = \begin{bmatrix}\boldsymbol{p}_1 & \cdots & \boldsymbol{p}_n\end{bmatrix}\begin{bmatrix}c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{bmatrix} = c_1\boldsymbol{p}_1 e^{\lambda_1 t} + \cdots + c_n \boldsymbol{p}_n e^{\lambda_n t}$
- Plugging the initial conditions $t = 0$ gives $c_1\boldsymbol{p}_1 + \cdots + c_n\boldsymbol{p}_n = \boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{c}$; solving the system gives the coefficients
- The eigenvalues λ are in the exponents, which dictate the speed at which the solution decays, or the frequency of oscillations
- The eigenvectors dictate the shape of the solution