Lecture 32, Apr 4, 2022

The Diagonalization Test

- Theorem VI: Diagonalization Test: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$ with distinct eigenvalues $\lambda_1, \cdots, \lambda_r$, then $\boldsymbol{A}$ is diagonalizable if and only if $\forall \alpha, m_\alpha = n_\alpha$, where m_α is the geometric multiplicity and n_α is the algebraic multiplicity
  - Proof: [$\implies$] Let $\boldsymbol{A}$ be diagonalizable, then:
    * If $\boldsymbol{A}$ is diagonalizable then there are n linearly independent eigenvectors; let $E = E_{\lambda_1} \cup \cdots \cup E_{\lambda_r}$ be a linearly independent set of eigenvectors where E_{λ_α} is a basis for each eigenspace
    * Since E is a basis for ${}^n\mathbb{R}$, we have $n = |E|$ where E is the cardinality of E (i.e. number of elements)
    * Since $E_{\lambda_i} \cap E_{\lambda_j} = \varnothing$, so then $n = |E| = \sum_{\alpha=1}^{r} |E_{\lambda_\alpha}| = \sum_{\alpha=1}^{r} m_\alpha \leq \sum_{\alpha=1}^{r} n_\alpha = n$
      - Note $n_1 + n_2 + \cdots + n_r = n$
    * Therefore $\sum_{\alpha=1}^{r} m_\alpha = \sum_{\alpha=1}^{r} n_\alpha$, and since $m_\alpha \leq n_\alpha$ we must have $m_\alpha = n_\alpha$ for all α
  - Proof: [$\impliedby$] For $m_\alpha = n\alpha, \forall \alpha$:
    * $|E| = \sum_{\alpha=1}^{r} |E_{\lambda_\alpha}| = \sum_{\alpha=1}^{r} m_\alpha = \sum_{\alpha=1}^{r} n_\alpha = n$
    * Since $|E| = n$ there are n linearly independent eigenvectors, which span and form a basis for ${}^n\mathbb{R}$, so $\boldsymbol{A}$ is diagonalizable