Lecture 31, Apr 1, 2022

Criteria for Diagonalizability

- $m_1 + m_2 + \cdots + m_r \leq n$ where $m_\alpha = \dim \mathscr{E}_{\lambda_\alpha}$ for $\boldsymbol{A} \in {}^n\mathbb{R}^n$
  - Proof: Let $\{ \boldsymbol{p}_{\alpha,j} \}$ be a set of eigenvectors in H_{λ_α}; $m_1 + \cdots + m_r$ is the total number of vectors in $H_{\lambda_1} \cup \cdots \cup H_{\lambda_\alpha}$; this cannot exceed n as that would violate the fundamental theorem
  - Corollary: If $m_1 + \cdots + m_r = n$, then $\boldsymbol{A}$ is diagonalizable
- Definition: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$ have eigenvalues λ_α; then the *algebraic multiplicity* of λ_α is n_α, the highest power of $(\lambda - \lambda_\alpha)$ that divides the characteristic equation for $\boldsymbol{A}$
  - i.e. the algebraic multiplicity of λ_α is the number of times λ_α appears as a root of the characteristic polynomial
- Definition: The *geometric multiplicity* of λ_α is $m_\alpha = \dim \mathscr{E}_{\lambda_\alpha}$, i.e. the dimension of its eigenspace
  - Theorem VI: Diagonalization Test: $m_\alpha = n_\alpha$ for all α if and only if the matrix is diagonalizable (proven next lecture)
- Proposition III: Let $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \boldsymbol{B} \\ \mathbf{0} & \boldsymbol{C} \end{bmatrix} \in {}^n\mathbb{R}^n$ and $\mathbf{1} \in {}^r\mathbb{R}^r$ (and $\boldsymbol{C} \in {}^{n-r}\mathbb{R}^{n-r}$), then $\det \boldsymbol{A} = \det \boldsymbol{C}$
  - Take $\boldsymbol{A}$ and reduce it into $\boldsymbol{A}' = \begin{bmatrix} \mathbf{1} & \boldsymbol{B}' \\ \mathbf{0} & \boldsymbol{C}' \end{bmatrix}$ where $\boldsymbol{C}$ is upper triangular
    * We can do this by some $\boldsymbol{E}$ without having to multiply any row by a scalar since we don't need the leading entries to be 1
  - $\det(\boldsymbol{A}') = \det(\boldsymbol{C}') = (-1)^p \det(\boldsymbol{C})$
  - $\det(\boldsymbol{A}') = (-1)^p \det(\boldsymbol{A})$ since the same operations were performed on $\boldsymbol{A}$
  - The minus signs cancel so $\det(\boldsymbol{A}) = \det(\boldsymbol{C})$
- Theorem V: Multiplicity Theorem: $1 \leq m_\alpha \leq n_\alpha$
  - Proof: Consider λ_α; Let $F = \{ \boldsymbol{f}_1, \cdots, \boldsymbol{f}_{m_\alpha} \}$ be a basis for $\mathscr{E}_{\lambda_\alpha}$ where $m_\alpha = \dim \mathscr{E}_{\lambda_\alpha}$
    * We can extend this basis for a basis for ${}^n\mathbb{R}$
    * Let $\boldsymbol{Q}$ be the transition matrix from F to $E_0 = \{ \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{m_\alpha}, \cdots, \boldsymbol{e}_n \}$, the standard basis for ${}^n\mathbb{R}$, then $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_n \end{bmatrix}$ and $\boldsymbol{Q}\boldsymbol{e}_\alpha = \boldsymbol{f}_\alpha$, i.e. $\boldsymbol{Q}^{-1}\boldsymbol{f}_\alpha = \boldsymbol{e}_\alpha$
    * Consider $\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}\boldsymbol{e}_{j_\alpha} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{f}_{j_\alpha} = \lambda_\alpha \boldsymbol{Q}^{-1}\boldsymbol{f}_{j\alpha} = \lambda_\alpha \boldsymbol{e}_{j_\alpha}$ where $j_\alpha = 1, \cdots, m_\alpha$
    * So $\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q} = \begin{bmatrix} \lambda_\alpha \mathbf{1} & \boldsymbol{B} \\ \mathbf{0} & \boldsymbol{C} \end{bmatrix}$ where $\mathbf{1} \in {}^{m_\alpha}\mathbb{R}^{m_\alpha}$ since using the standard basis vectors we can pick out the λ_α for the first m_α columns
    * Consider
      $$\begin{aligned} c_{\boldsymbol{A}}(\lambda) &= c_{\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}}(\lambda) \\ &= \det \begin{bmatrix} (\lambda - \lambda_\alpha)\mathbf{1} & -\boldsymbol{B} \\ \mathbf{0} & \lambda\mathbf{1} - \boldsymbol{C} \end{bmatrix} \\ &= (\lambda - \lambda_\alpha)^{m_\alpha} \det \begin{bmatrix} \mathbf{1} & -\boldsymbol{B} \\ \mathbf{0} & \lambda\mathbf{1} - \boldsymbol{C} \end{bmatrix} \\ &= (\lambda - \lambda_\alpha)^{m_\alpha} \det(\lambda\mathbf{1} - \boldsymbol{C}) \end{aligned}$$
      - Note the first line relies on the similarity transformation preserving the characteristic equation
      - The last line relies on Prop. III
    * This shows us that we have to have at least m_α repeated roots of λ_α, so $m_\alpha \leq n_\alpha$
    * Since every eigenspace must have at least one nontrivial eigenvector $m_\alpha \geq 1$