Lecture 30, Mar 29, 2022

Independence of Eigenspaces

- Theorem III: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$ have r distinct eigenvalues ($r \leq n$) denoted $\lambda_1, \cdots, \lambda_r$, and let $\boldsymbol{x}_\alpha \in \mathscr{E}_{\lambda_\alpha}$ but $\boldsymbol{x}_\alpha \neq \boldsymbol{0}$; then $\{ \boldsymbol{x}_1, \cdots, \boldsymbol{x}_r \}$ is linearly independent
  - i.e. eigenvectors corresponding to different eigenvalues are always linearly independent
  - Proof by induction:
    * For $k = 1$, the set $\{ \boldsymbol{x}_1 \}$ is linearly independent
    * Assume $\{ \boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \}$ is linearly independent
    * Consider

$$
\begin{aligned}
& \sum_{i=1}^{k+1} \mu_i \boldsymbol{x}_i = \boldsymbol{0} \\
\implies & \sum_{i=1}^{k+1} \mu_i \boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{0} \\
\implies & \sum_{i=1}^{k+1} \mu_i \lambda_i \boldsymbol{x}_i = \boldsymbol{0} \\
\implies & \sum_{i=1}^{k+1} \mu_i \lambda_i \boldsymbol{x}_i - \lambda_{k+1} \sum_{i=1}^{k+1} \mu_i \boldsymbol{x}_i = \boldsymbol{0} \\
\implies & \sum_{i=1}^{k+1} \mu_i (\lambda_i - \lambda_{k+1}) \boldsymbol{x}_i = \boldsymbol{0} \\
\implies & \sum_{i=1}^{k} \mu_i (\lambda_i - \lambda_{k+1}) \boldsymbol{x}_i = \boldsymbol{0} \\
\implies & \mu_1, \cdots, \mu_k = \boldsymbol{0} \\
\implies & \mu_{k+1} \boldsymbol{x}_{k+1} = \boldsymbol{0} \\
\implies & \mu_{k+1} = 0 \\
\implies & \{ \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{k+1} \} \text{ is linearly independent}
\end{aligned}
$$

  - Corollary: If all the eigenvalues of $\boldsymbol{A}$ are distinct, then $\boldsymbol{A}$ is diagonalizable (since if $r = n$, we can pick a set of n independent eigenvectors, which must be a basis for ${}^n\mathbb{R}$)
    * However, if the eigenvalues are not distinct, that doesn't mean the matrix is not diagonalizable
- Lemma I: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$ have r distinct eigenvalues and $\boldsymbol{x}_\alpha \in \mathscr{E}_{\lambda_\alpha}$, if $\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n = \boldsymbol{0}$ then $\boldsymbol{x}_\alpha = \boldsymbol{0}$
  - Proof:
    * Consider $\mu_1 \boldsymbol{x}_1 + \cdots + \mu_r \boldsymbol{x}_r = \boldsymbol{0}$
    * If $\boldsymbol{x}_\alpha \neq 0$ for some α then $\mu_\alpha = 0$ since $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r$ are independent
    * This contradicts $\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_r = \boldsymbol{0}$
- Theorem IV: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$ have r distinct eigenvalues and H_{λ_α} be a linearly independent set of eigenvectors from $\mathscr{E}_{\lambda_\alpha}$, then $H_{\lambda_1} \cup H_{\lambda_2} \cup \cdots \cup H_{\lambda_r}$ is linearly independent
  - Proof:
    * Let $H_{\lambda_\alpha} = \{ \boldsymbol{p}_{\alpha,1}, \cdots, \boldsymbol{p}_{\alpha,m_\alpha} \}$
    * $\displaystyle\sum_{j=1}^{m_1} \mu_{1,j} \boldsymbol{p}_{1,j} + \sum_{j=2}^{m_2} \mu_{2,j} \boldsymbol{p}_{2,j} + \cdots + \sum_{j=1}^{m_r} \mu_{r,j} \boldsymbol{p}_{r,j} = \boldsymbol{0}$
    * $\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_r = 0 \implies \boldsymbol{x}_\alpha = \boldsymbol{0}$ by Lemma I
    * Since each sum adds up to zero, all μ are zero since each H is linearly independent
    * Therefore the union of all the sets is linearly independent