Lecture 29, Mar 28, 2022

Diagonalization Properties

- Proposition II: Let $\boldsymbol{A}, \boldsymbol{T} \in {}^n\mathbb{R}^n$ and $\boldsymbol{T}$ be invertible, then $\boldsymbol{A}$ and $\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$ have the same characteristic polynomial and therefore same eigenvalues
  - $\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$ is known as a *similarity transformation* of $\boldsymbol{A}$
  - Proof:
$$\begin{aligned}&\det(\lambda\mathbf{1} - \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T})\\ &= \det(\lambda\boldsymbol{T}^{-1}\boldsymbol{T} - \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T})\\ &= \det(\boldsymbol{T}^{-1}(\lambda\mathbf{1} - \boldsymbol{A})\boldsymbol{T})\\ &= \det(\boldsymbol{T}^{-1})\det(\lambda\mathbf{1} - \boldsymbol{A})\det(\boldsymbol{T})\\ &= \det(\lambda\mathbf{1} - \boldsymbol{A})\end{aligned}$$
- Theorem II: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$ be diagonalizable; then:
  1. The characteristic equation for $\boldsymbol{A}$ can be written as $c_{\boldsymbol{A}}(\lambda) = \det(\lambda\mathbf{1} - \boldsymbol{A}) = \prod_{\alpha=1}^{n}(\lambda - \lambda_\alpha)$
     - If $\boldsymbol{A}$ is diagonalized by $\boldsymbol{P}$ then $c_{\boldsymbol{A}}(\lambda) = c_{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}}(\lambda) = c_{\boldsymbol{\Lambda}}(\lambda)$
  2. $\det(\boldsymbol{A}) = \prod_{\alpha=1}^{n}\lambda_\alpha$
     - Proof: $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} \implies$
$$\begin{aligned}\det(\boldsymbol{A}) &= \det(\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1})\\ &= \det(\boldsymbol{P})\det(\boldsymbol{\Lambda})\det(\boldsymbol{P}^{-1})\\ &= \det(\boldsymbol{\Lambda})\\ &= \prod_{\alpha=1}^{n}(\lambda - \lambda_\alpha)\end{aligned}$$
  3. $\operatorname{tr}\boldsymbol{A} = \sum_{\alpha=1}^{n}\lambda_\alpha$
     - Proof: $\operatorname{tr}\boldsymbol{A} = \operatorname{tr}(\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}) = \operatorname{tr}(\boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{\Lambda}) = \operatorname{tr}(\boldsymbol{\Lambda}) = \sum_{\alpha=1}^{n}\lambda_\alpha$
     - Note $\operatorname{tr}(\boldsymbol{S}\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{T}\boldsymbol{S})$
- Theorem II holds for all matrices, even ones that are not diagonalizable, we just currently cannot prove it
- It's important to note that repeated eigenvalues are counted multiple times