Lecture 28, Mar 25, 2022

Properties of Eigenvalues and Eigenvectors

- If $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the eigenvalues satisfy $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - (\operatorname{tr} \boldsymbol{A})\lambda + \det \boldsymbol{A} = 0$
- In general, for $\boldsymbol{A} \in {}^n\mathbb{R}^n$, $\lambda^n - (\operatorname{tr} \boldsymbol{A})\lambda^{n-1} + \cdots + (-1)^n \det \boldsymbol{A} = 0$
- Proposition I: Let λ, μ be two distinct eigenvalues for $\boldsymbol{A} \in {}^n\mathbb{R}^n$, then $\mathscr{E}_\lambda \cap \mathscr{E}_\mu = \{ \boldsymbol{0} \}$
  - Proof: Let $\boldsymbol{x} \in \mathscr{E}_\lambda \cap \mathscr{E}_\mu$, then $\boldsymbol{Ax} = \lambda\boldsymbol{x}$ and $\boldsymbol{Ax} = \mu\boldsymbol{x} \implies \lambda\boldsymbol{x} = \mu\boldsymbol{x} \implies (\lambda - \mu)\boldsymbol{x} = \boldsymbol{0}$, but $\lambda \neq \mu$ so $\boldsymbol{x} = \boldsymbol{0}$
- Note eigenvalues may be complex

Diagonalizability

- We have $\begin{cases} \boldsymbol{Ap}_1 = \lambda_1\boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{Ap}_n = \lambda_n\boldsymbol{p}_n \end{cases} \implies \boldsymbol{A}\begin{bmatrix}\boldsymbol{p}_1 & \cdots & \boldsymbol{p}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{p}_1 & \cdots & \boldsymbol{p}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$
  - This can be written as $\boldsymbol{AP} = \boldsymbol{P\Lambda}$
  - Suppose $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n$ are linearly independent, then $\boldsymbol{P}$ is invertible, then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{\Lambda}$
    - This is called *diagonalization* since $\boldsymbol{\Lambda}$ is a diagonal matrix
- If we had $\dot{\boldsymbol{\eta}} = \boldsymbol{\Lambda\eta} \implies \{ \dot{\eta}_i = \lambda_i\eta_i \}$, which is a system of decoupled differential equations
- Definition: $\boldsymbol{P} \in {}^n\mathbb{R}^n$ *diagonalizes* $\boldsymbol{A} \in {}^n\mathbb{R}^n$ if $\boldsymbol{P}$ is invertible and $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{\Lambda}$
  - However, it's not always possible to find a set of $\boldsymbol{p}_n$ such that $\boldsymbol{P}$ is invertible, i.e. not all matrices are diagonalizable
- Theorem I: Diagonalization Theorem: The matrix $\boldsymbol{P} \in {}^n\mathbb{R}^n$ diagonalizes $\boldsymbol{A} \in {}^n\mathbb{R}^n$ (i.e. $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{\Lambda}$) iff $\boldsymbol{P}$ the columns of $\boldsymbol{P}$ are eigenvectors of $\boldsymbol{A}$ that form a basis for ${}^n\mathbb{R}$
- Note that $P^{-1}\boldsymbol{AP} = \boldsymbol{\Lambda} \implies \boldsymbol{A} = \boldsymbol{P\Lambda P}^{-1}$