Lecture 27, Mar 22, 2022

Eigenvalues and Eigenvectors: Definition and Motivation

- Motivation: Finding solutions to a system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x} = \boldsymbol{x}(t) \in {}^n\mathbb{R}$, where the dot indicates time derivative
  - Assume that $\boldsymbol{A} \in {}^n\mathbb{R}^n$ is constant
  - Each equation is first order, but higher order equations can also be expressed in this form by making derivatives also variables
- As in the case for scalars, try $\boldsymbol{x}(t) = \boldsymbol{p}e^{\lambda t}$ where $\boldsymbol{p} \in {}^n\mathbb{R}$
  - $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \implies \lambda\boldsymbol{p}e^{\lambda t} = \boldsymbol{A}\boldsymbol{p}e^{\lambda t} \implies \lambda p = \boldsymbol{A}\boldsymbol{p} \implies (\lambda\mathbf{1} - \boldsymbol{A})\boldsymbol{p} = \mathbf{0}$
    * This is similar to the characteristic equation in the scalar case
    * We can say that $\boldsymbol{p} \neq \mathbf{0}$ since that would give the trivial solution
    * This means that $(\lambda\mathbf{1} - \boldsymbol{A})$ must have a null space, which means $\lambda\mathbf{1} - \boldsymbol{A}$ cannot have full rank, so we must choose λ such that $(\lambda\mathbf{1} - \boldsymbol{A})$ is singular, i.e. $\det(\lambda\mathbf{1} - \lambda A) = 0$
    * The "eigenproblem"
  - The λ that make $\det(\lambda\mathbf{1} - \lambda\boldsymbol{A}) = 0$ are the *eigenvalues* of $\boldsymbol{A}$
  - The nontrivial $\boldsymbol{p}$ are the *eigenvectors* (for a particular λ)
    * Note these can be scaled arbitrarily
- For $\boldsymbol{A} \in {}^n\mathbb{R}^n$, there are n such λ, because $\det(\lambda\mathbf{1} - \lambda\boldsymbol{A})$ is an n-th degree polynomial of λ
  - Expanding out the determinant, we obtain the *characteristic polynomial* (eigenpolynomial?) of this system of differential equations; when we set it to zero, we obtain the *characteristic equation* (eigenequation?)
  - Notation: $C_{\boldsymbol{A}}(\lambda)$ for the eigenpolynomial
- Since $\boldsymbol{p} \in \text{null}(\lambda\mathbf{1} - \boldsymbol{A})$, the *eigenspace* for an eigenvalue λ is $\{\, \boldsymbol{p} \in {}^n\mathbb{R} \mid \boldsymbol{A}\boldsymbol{p} = \lambda\boldsymbol{p} \,\} = \text{null}(\lambda\mathbf{1} - \lambda\boldsymbol{A})$ (sometimes denoted $\mathscr{E}_\lambda$)
  - The bases for the eigenspaces are the eigenvectors
  - All the eigenvectors are linearly independent
  - Note that since $\mathbf{0}$ is the trivial eigenvector, normally we use "eigenvector" to refer to only nonzero eigenvectors
- If $\boldsymbol{A}$ is viewed as a linear transformation, eigenvectors are the vectors that are scaled by the transformation by an eigenvalue (i.e. direction remains unchanged)
- This allows us to solve the general n-th order differential equation
  - Let $x_1 = x, x_2 = \dot{x}$, then $\dot{x}_1 = x_2, \dot{x}_2 = \ddot{x} = -a_1\dot{x} - a_0x = -a_1x_2 - a_0x_1$
  - We can put this in a matrix as $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$
  - By extension this can be used to solve a linear system of any order
- The eigenvalues of an upper triangular matrix are the values on the diagonal of the matrix (since the determinant of such a matrix is the product of the diagonal)
- "Eigen" is a German word meaning "characteristic, proper"