Lecture 25, Mar 18, 2022

Cramer's Rule

- Cramer's Rule (Maclaurin-Cramer Rule): The solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} \in {}^n\mathbb{R}^n$ is given by $x_i = \dfrac{\det \boldsymbol{A}_i}{\det \boldsymbol{A}}$ where $\boldsymbol{x}_i$ are the components of $\boldsymbol{x}$ and $\boldsymbol{A}_i$ is $\boldsymbol{A}$ with column i replaced by $\boldsymbol{b}$, if $\det \boldsymbol{A} \neq 0$
  - $\boldsymbol{A}_i = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{b} & \cdots & \boldsymbol{c}_n \end{bmatrix}$
  - $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n \end{bmatrix} \boldsymbol{x} = \displaystyle\sum_{j=1}^{n} x_j \boldsymbol{c}_j$
  - $$\begin{aligned} \det \boldsymbol{A}_i &= \det \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \displaystyle\sum_{j=1}^{n} x_j \boldsymbol{c}_j & \cdots & \boldsymbol{c}_n \end{bmatrix} \\ &= \det \begin{bmatrix} \boldsymbol{c}_1 & \cdots & x_i \boldsymbol{c}_i + \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j \boldsymbol{c}_j & \cdots & \boldsymbol{c}_n \end{bmatrix} \\ &= \det \begin{bmatrix} \boldsymbol{c}_1 & \cdots & x_i \boldsymbol{c}_i & \cdots & \boldsymbol{c}_n \end{bmatrix} \\ &= x_i \det \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_i & \cdots & \boldsymbol{c}_n \end{bmatrix} \\ &= x_i \det \boldsymbol{A} \end{aligned}$$
  - Provided that $\det \boldsymbol{A} \neq 0$, we have $\det \boldsymbol{A}_i = x_i \det \boldsymbol{A} \implies x_i = \dfrac{\det \boldsymbol{A}_i}{\det \boldsymbol{A}}$
  - Note the sum is just adding multiples of other columns, which does not change the determinant as per determinant properties
- Cramer's rule is computationally inefficient for larger matrices ($\mathcal{O}((n+1)!)$ operations if taking determinants recursively); Gaussian elimination is much better for larger matrices ($\mathcal{O}(n^3)$ operations)
  - For smaller matrices it might be faster; in general this depends on the matrix itself (e.g. number of zeros)

Cofactors and Adjoints

- Definition: The (i, j) cofactor of $\boldsymbol{A} \in {}^n\mathbb{R}^n$ is $c_{ij}(\boldsymbol{A}) = (-1)^{i+j} \det \boldsymbol{M}_{ij}(\boldsymbol{A})$
  - Using the cofactor, the determinant can be written as $\displaystyle\sum_{j=1}^{n} a_{kj} c_{kj}$, true for any k
- $\displaystyle\sum_{j=1}^{n} a_{ij} c_{kj} = \begin{cases} \det \boldsymbol{A} & k = i \\ 0 & k \neq i \end{cases}$
  - Proof: Consider $\boldsymbol{A}' \in {}^n\mathbb{R}^n$ which is $\boldsymbol{A}$ with row k replaced with row i
    - $\det \boldsymbol{A}' = 0$ because rows are not independent
    - Using the Laplace expansion about row k: $\det \boldsymbol{A}' = 0 = \displaystyle\sum_{j=1}^{n} a'_{kj} c_{kj}(\boldsymbol{A}') = \sum_{j=1}^{n} a_{ij} c_{kj}(\boldsymbol{A})$
      - $a'_{kj} = a_{ij}$ because we replaced row k by row i
      - $c_{kj}(\boldsymbol{A}') = c_{kj}(\boldsymbol{A})$ because row k was eliminated in the calculation of the cofactor so the minors are the same
- $\displaystyle\sum_{j=1}^{n} a_{ij} c_{kj}$ is like taking $\boldsymbol{A}\boldsymbol{C}^T$ where $\boldsymbol{C} = \begin{bmatrix} c_{kj} \end{bmatrix}$ is the cofactor matrix
- $\begin{cases} \det \boldsymbol{A} & k = i \\ 0 & k \neq i \end{cases}$ is just $(\det \boldsymbol{A})\mathbf{1}$
- $\boldsymbol{A}\boldsymbol{C}^T = (\det \boldsymbol{A})\mathbf{1}$
- Definition: The *adjoint* of $\boldsymbol{A}$ is $\operatorname{adj} \boldsymbol{A} = \boldsymbol{C}^T$
  - Also known as the *adjugate*
- Theorem VIII: $\boldsymbol{A}(\operatorname{adj} \boldsymbol{A}) = (\det \boldsymbol{A})\mathbf{1} = (\operatorname{adj} \boldsymbol{A})\boldsymbol{A}$

  - If $\boldsymbol{A}$ is invertible then $\boldsymbol{A}^{-1} = \dfrac{1}{\det \boldsymbol{A}} \operatorname{adj} \boldsymbol{A}$
- $\det(\operatorname{adj} \boldsymbol{A}) = (\det \boldsymbol{A})^{n-1}$
- If $\boldsymbol{A}$ is non-invertible, then $(\operatorname{adj} \boldsymbol{A})\boldsymbol{A} = \mathbf{0} \implies \operatorname{col} \boldsymbol{A} \subseteq \operatorname{null} \operatorname{adj} \boldsymbol{A}$; if $\boldsymbol{A} \neq \mathbf{0}$, $\dim \operatorname{col} \boldsymbol{A} \geq 1 \implies \dim \operatorname{null} \operatorname{adj} \boldsymbol{A} \geq 1$
  - $n - \operatorname{rank} \operatorname{adj} \boldsymbol{A} = \dim \operatorname{null} \operatorname{adj} \boldsymbol{A} \implies \operatorname{rank} \operatorname{adj} \boldsymbol{A} < n \implies \operatorname{adj} \boldsymbol{A}$ is not invertible
  - $\det(\operatorname{adj} \boldsymbol{A}) = 0$ so the previous equation still holds