Lecture 24, Mar 15, 2022

Additional Properties of the Determinant

- Determinant of elementary matrices: $\det \boldsymbol{E}(i,j) = -1$, $\det \boldsymbol{E}(\lambda; i) = \lambda$, $\det \boldsymbol{E}(\lambda; i, j) = 1$
- Theorem IV: Cauchy-Binet Product Rule: Let $\boldsymbol{A}, \boldsymbol{B} \in {}^n\mathbb{R}^n$, then $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
  - Proof: Define $\Delta_{\boldsymbol{B}}(\boldsymbol{A}) = \det(\boldsymbol{AB})$
    * Claim: $\Delta_{\boldsymbol{B}}$ is a proper determinant function:
      1. $\Delta_{\boldsymbol{B}}\left[\boldsymbol{E}(1; i, j)\boldsymbol{A}\right] = \det\left[\boldsymbol{E}(1; i, j)\boldsymbol{AB}\right] = \det(\boldsymbol{AB}) = \Delta_{\boldsymbol{B}}(\boldsymbol{A})$
      2. $\Delta_{\boldsymbol{B}}\left[\boldsymbol{E}(\lambda; i)\boldsymbol{A}\right] = \det\left[\boldsymbol{E}(\lambda; i)\boldsymbol{AB}\right] = \lambda\det(\boldsymbol{AB}) = \lambda\Delta_{\boldsymbol{B}}(\boldsymbol{A})$
    * From $\Delta_n(\boldsymbol{A}) = \det(\boldsymbol{A})\Delta_n(\mathbf{1})$ we know $\Delta_{\boldsymbol{B}}(\boldsymbol{A}) = \det(\boldsymbol{A})\Delta_{\boldsymbol{B}}(\mathbf{1}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
    * Therefore $\det(\boldsymbol{A})\det(\boldsymbol{B}) = \Delta_{\boldsymbol{B}}(\boldsymbol{A}) = \det(\boldsymbol{AB})$
- Theorem V: Transpose Rule: Let $\boldsymbol{A} \in {}^n\mathbb{R}^n$, then $\det \boldsymbol{A} = \det \boldsymbol{A}^T$
  - This means we can also compute the determinant along rows instead of columns, since the matrix can be transposed and the determinant is unchanged
  - Proof: Define $\Delta_T(\boldsymbol{A}) = \det \boldsymbol{A}^T$
    * Claim: Δ_T is a proper determinant function:
      1. $\Delta_T\left[\boldsymbol{E}(1; i, j)\boldsymbol{A}\right] = \det(\boldsymbol{EA})^T = \det(\boldsymbol{A}^T\boldsymbol{E}^T) = \det(\boldsymbol{A}^T)\det(\boldsymbol{E}^T) = \det(\boldsymbol{A}^T)\det(\boldsymbol{E}(\lambda; j, i)) = \det(\boldsymbol{A}^T) = \Delta_T(\boldsymbol{A})$
      2. $\Delta_T\left[\boldsymbol{E}(\lambda; i)\boldsymbol{A}\right] = \det(\boldsymbol{A}^T)\det(\boldsymbol{E}^T) = \det(\boldsymbol{A}^T)\det(\boldsymbol{E}(\lambda; i)) = \lambda\det(\boldsymbol{A}^T) = \lambda\Delta_T(\boldsymbol{A})$
      3. $\Delta_T(\mathbf{1}) = \det(\mathbf{1}^T) = \det \mathbf{1} = 1$
    * Therefore Δ_T is the determinant, and since the determinant is unique, $\det A^T = \Delta_T(\boldsymbol{A}) = \det \boldsymbol{A}$
- Theorem VI: Invertibility theorem: $\boldsymbol{A} \in {}^n\mathbb{R}^n$ is invertible iff $\det \boldsymbol{A} \neq 0$
  - Proof:
    * If $\boldsymbol{A}$ invertible, then $\boldsymbol{AA}^{-1} = \mathbf{1} \implies \det(\boldsymbol{AA}^{-1}) = \det \mathbf{1} = 1 \implies \det(\boldsymbol{A})\det(\boldsymbol{A}^{-1}) = 1$ so $\det(\boldsymbol{A}) \neq 0$
      - Corollary: $\det \boldsymbol{A}^{-1} = \dfrac{1}{\det \boldsymbol{A}}$ if $\boldsymbol{A}$ is invertible
    * By contraposition, if $\boldsymbol{A}$ is not invertible, then its rows are dependent, then $\det \boldsymbol{A} = 0$; therefore $\det \boldsymbol{A} \neq 0 \implies \boldsymbol{A}$ is invertible