Lecture 23, Mar 14, 2022

The Determinant

- Lemma I: If $\Delta_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ is a determinant function, then $\Delta_n(\boldsymbol{A}) = \kappa(\boldsymbol{A})\Delta_n(\mathbf{1})$ where $\kappa(\boldsymbol{A})$ is a scalar function of $\boldsymbol{A}$
  - Proof: Gaussian eliminate on $\boldsymbol{A}$ until it is upper triangular, i.e. $\boldsymbol{E}_1 \cdots \boldsymbol{E}_n\boldsymbol{U}$; $\Delta_n(\boldsymbol{U}) = \Delta_n(\mathbf{1})\prod_{i=1}^{n} u_{ii}$ which is a scalar times $\Delta_n(\mathbf{1})$; since elementary matrices either scale, negate, or leave the determinant unchanged, the final result is going to be a scalar times $\Delta_n(\mathbf{1})$
- Theorem II: Let $\Delta_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ and $\hat{\Delta}_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$; $\hat{\Delta}_n$ satisfies an additional property DIII $\hat{\Delta}_n(\mathbf{1}) = 1$; then $\Delta_n(\boldsymbol{A}) = \hat{\Delta}_n(\boldsymbol{A})\Delta_n(\mathbf{1})$
  - Proof: Consider $\begin{aligned}[t] \Delta_n(\boldsymbol{A}) - \hat{\Delta}_n(\boldsymbol{A})\Delta_n(\mathbf{1}) &= \kappa(\boldsymbol{A})\Delta_n(\mathbf{1}) - \kappa(\boldsymbol{A})\hat{\Delta}_n(\mathbf{1})\Delta_n(\mathbf{1}) \\ &= \kappa(\boldsymbol{A})\Delta_n(\mathbf{1}) - \kappa(\boldsymbol{A})\Delta_n(\mathbf{1}) \\ &= 0 \end{aligned}$
  - Corollary: If $\Delta_n(\mathbf{1}) = 1$ as well, then $\Delta_n(\boldsymbol{A}) = \hat{\Delta}_n(\boldsymbol{A})$
    - If DIII also holds, then the determinant function is unique
- Definition: The *determinant* of $\boldsymbol{A} \in {}^n\mathbb{R}^n$ is the unique determinant function $\Delta_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ that satisfies:
  1. DI: Adding one row to another row leaves the result unchanged: $\Delta_n\left[\boldsymbol{E}(1; i, j)\boldsymbol{A}\right] = \Delta_n(\boldsymbol{A})$
     - $\boldsymbol{E}(\lambda; i, j)$ is an elementary matrix of type III that multiplies row j by λ and adds it to row i
  2. DII: $\Delta_n\left[\boldsymbol{E}(\lambda, i)\boldsymbol{A}\right] = \lambda\Delta_n(\boldsymbol{A})$
     - $\boldsymbol{E}(\lambda, i)$ is an elementary matrix of type II that multiplies row i by λ
  3. DIII: $\Delta_n(\mathbf{1}) = 1$
- Definition: The (i, j) *minor* of a square matrix $\boldsymbol{A} \in {}^n\mathbb{R}^n$, denoted $\boldsymbol{M}_{ij}(\boldsymbol{A}) \in {}^{n-1}\mathbb{R}^{n-1}$, is the matrix obtained by eliminating the i-th row and j-th column
- Definition: The function $\det_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ is $\det_n\boldsymbol{A} = \sum_{k=1}^{n}(-1)^{k+j}a_{kj}\det_{n-1}\boldsymbol{M}_{kj}(\boldsymbol{A})$ for any $1 \leq j \leq n$ and $\det_1\left[a\right] = a$
  - We can use any column and the definition still works
  - This is known as the *Laplace expansion*
- Theorem III: $\det_n {}^n\mathbb{R}^n \mapsto \mathbb{R}$ is the determinant
  - This shows the existence and uniqueness of the determinant
- The determinant can also be denoted $|\boldsymbol{A}|$