Lecture 21, Mar 8, 2022

The Determinant Function

- Every matrix has a determinant denoted $\det(\boldsymbol{A})$; for 2×2 matrix this is $ad - bc$
- Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_n \end{bmatrix} \in {}^n\mathbb{R}^n$; then the *determinant function* $\Delta_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ is any function that satisfies the following:
  1. Adding one row to another row leaves the result unchanged: $\Delta_n\left[\boldsymbol{E}(1; i, j)\boldsymbol{A}\right] = \Delta_n(\boldsymbol{A})$
     - $\boldsymbol{E}(\lambda; i, j)$ is an elementary matrix of type III that multiplies row j by λ and adds it to row i
  2. $\Delta_n\left[\boldsymbol{E}(\lambda, i)\boldsymbol{A}\right] = \lambda\Delta_n(\boldsymbol{A})$
     - $\boldsymbol{E}(\lambda, i)$ is an elementary matrix of type II that multiplies row i by λ
- The determinant function is homogeneous in each row (i.e. scaling a row scales the entire determinant)
- Theorem I: $\Delta_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ has the properties:
  1. If $\boldsymbol{A}$ has a zero row, then $\Delta_n(\boldsymbol{A}) = 0$
     - Proof: If row i of $\boldsymbol{A}$ is zero, then $\boldsymbol{E}(0, i)\boldsymbol{A} = \boldsymbol{A}$, therefore $\Delta_n\left[\boldsymbol{E}(0, i)\boldsymbol{A}\right] = \Delta_n(\boldsymbol{A}) = 0\Delta_n(\boldsymbol{A}) = 0$
  2. $\Delta_n\left[\boldsymbol{E}(\lambda; i, j)\boldsymbol{A}\right] = \Delta_n(\boldsymbol{A})$ (property 1, but with any scalar multiple)
     - Proof: Trivially true for $\lambda = 0$; for nonzero λ, scale row j by λ (scales the determinant by j), then add row j to row i (determinant unchanged), then divide the result by λ (determent scaled by $\frac{1}{\lambda}$), which gives the same determinant
  3. Interchanging rows negates the determinant: $\Delta_n\left[\boldsymbol{E}(i, j)\boldsymbol{A}\right] = -\Delta_n(\boldsymbol{A})$
     - Proof: $\Delta_n \begin{bmatrix} \vdots \\ \boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_i \\ \vdots \end{bmatrix} = -\Delta_n \begin{bmatrix} \vdots \\ -\boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_i \\ \vdots \end{bmatrix} = -\Delta_n \begin{bmatrix} \vdots \\ -\boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_i - \boldsymbol{r}_j \\ \vdots \end{bmatrix} = \Delta_n \begin{bmatrix} \vdots \\ -\boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_j - \boldsymbol{r}_i \\ \vdots \end{bmatrix} = \Delta_n \begin{bmatrix} \vdots \\ -\boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_j - \boldsymbol{r}_i \\ \vdots \end{bmatrix} = -\Delta_n \begin{bmatrix} \vdots \\ \boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_j - \boldsymbol{r}_i \\ \vdots \end{bmatrix} = -\Delta_n \begin{bmatrix} \vdots \\ \boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_j \\ \vdots \end{bmatrix}$
  4. If the rows of $\boldsymbol{A}$ are linearly dependent then $\Delta_n(\boldsymbol{A}) = 0$
     - Proof: If the rows are dependent, then at least one row can be written as a linear combination of the others; therefore by adding multiples of other rows to this row (does not change the determinant by property 2), it is possible to make this row all zero, which means the determinant is 0 (by property 1)
  5. The determinant function is linear in every row (n-linear): $\Delta_n \begin{bmatrix} \vdots \\ \lambda\boldsymbol{p} + \mu\boldsymbol{q} \\ \vdots \end{bmatrix} = \lambda\Delta_n \begin{bmatrix} \vdots \\ \boldsymbol{p} \\ \vdots \end{bmatrix} + \mu\Delta_n \begin{bmatrix} \vdots \\ \boldsymbol{q} \\ \vdots \end{bmatrix}$
     - Proof: Without loss of generality, show $\Delta_n \begin{bmatrix} \boldsymbol{p} + \boldsymbol{q} \\ \vdots \end{bmatrix} = \Delta_n \begin{bmatrix} \boldsymbol{p} \\ \vdots \end{bmatrix} + \Delta_n \begin{bmatrix} \boldsymbol{q} \\ \vdots \end{bmatrix}$
       * If the rest of the rows are dependent, then by property 4 each determinant is 0 so $0 = 0 + 0$
       * If the rest of the rows are independent, extend the rest of the rows to a basis for $\mathbb{R}^n$ by adding an independent vector; then $\boldsymbol{p} = \sum_{i=1}^{n} \lambda_i \boldsymbol{r}_i$ and $\boldsymbol{q} = \sum_{i=1}^{n} \mu_k \boldsymbol{r}_k$ so $\Delta_n \begin{bmatrix} \boldsymbol{p} + \boldsymbol{q} \\ \vdots \end{bmatrix} =$

$\Delta_n \begin{bmatrix} (\lambda_1+\mu_1)\boldsymbol{r}_1 + \sum_{k=2}^{n}(\lambda_k+\mu_k)\boldsymbol{r}_k \\ \vdots \end{bmatrix} = \Delta_n \begin{bmatrix} (\lambda_1+\mu_1)\boldsymbol{r}_1 \\ \vdots \end{bmatrix} = (\lambda_1+\mu_1)\Delta_n \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \end{bmatrix}$; also by the same process $\Delta_n \begin{bmatrix} \boldsymbol{p} \\ \vdots \end{bmatrix} = \lambda_1 \Delta_n \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \end{bmatrix}$ and $\Delta_n \begin{bmatrix} \boldsymbol{q} \\ \vdots \end{bmatrix} = \mu_1 \Delta_n \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \end{bmatrix}$