Lecture 20, Mar 7, 2022

Change of Basis

- Say we have a vector expressed in a vector space as coordinates with respect to one set of basis vectors; how do we express them in terms of another basis?
- The new coordinates in terms of the new basis is related to the old coordinates by a matrix
- $\boldsymbol{v} = \boldsymbol{P}\boldsymbol{v}'$ where $\boldsymbol{P}$ is the *transition matrix* or *transformation matrix* or *change-of-basis matrix*
- Let $\mathcal{V}$ where $\dim \mathcal{V} = n$ have 2 bases: $E = \{ \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n \}$ and $F = \{ \boldsymbol{f}_1, \cdots, \boldsymbol{f}_n \}$
  - $\boldsymbol{v} = \sum_{i=1}^{n} v_i^{(e)} \boldsymbol{e}_i = \sum_{i=1}^{n} v_i^{(f)} \boldsymbol{f}_i$
  - Let $\boldsymbol{v}_e = \begin{bmatrix} v_1^{(e)} \\ \vdots \\ v_n^{(e)} \end{bmatrix}, \boldsymbol{v}_f = \begin{bmatrix} v_1^{(f)} \\ \vdots \\ v_n^{(f)} \end{bmatrix}$
    - $\boldsymbol{v}_e$ are the coordinates in terms of E, $\boldsymbol{v}_f$ are the coordinates in terms of F
  - What is the relationship between $\boldsymbol{v}_e$ and $\boldsymbol{v}_f$?
- Since the bases live in $\mathcal{V}$, in general $\boldsymbol{e}_j = \sum_{i=1}^{n} p_{ij} \boldsymbol{f}_i$
  - $$\begin{aligned} \boldsymbol{v} &= \sum_{i=1}^{n} v_i^{(f)} \boldsymbol{f}_i \\ &= \sum_{j=1}^{n} v_j^{(e)} \boldsymbol{e}_j \\ &= \sum_{j=1}^{n} v_j^{(e)} \left(\sum_{i=1}^{n} p_{ij} \boldsymbol{f}_i \right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij} v_j^{(e)} \right) \boldsymbol{f}_i \end{aligned}$$
  - $v_i^{(f)} = \sum_{j=1}^{n} p_{ij} v_j^{(e)}$ or $\boldsymbol{v}_f = \boldsymbol{P}\boldsymbol{v}_e$
  - The *columns* of $\boldsymbol{P}$ are the basis $\boldsymbol{e}_n$ expressed in terms of $\boldsymbol{f}_n$
- Proposition III: Let $B_e = \{ \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n \}$ be the standard basis for ${}^n\mathbb{R}$ and $B_f = \{ \boldsymbol{f}_1, \cdots \boldsymbol{f}_n \}$ be another basis; then the transition matrix from B_f to B_e is $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_n \end{bmatrix}$
- There is an isomorphism between $\mathcal{V} \leftrightarrow {}^n\mathbb{R}$; instead of thinking of members of $\mathcal{V}$ directly, we can think about their coordinates, which are vectors in ${}^n\mathbb{R}$
- Theorem II: Let $B_e = \{ \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n \} \subset \mathcal{V}$ be a basis for $\mathcal{V}$ and the coordinates $\boldsymbol{v}' \in {}^n\mathbb{R}$ for $\boldsymbol{v} \in \mathcal{V}$; then $\{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_m \}$ is linearly independent iff the coordinates $\{ \boldsymbol{v}_1', \cdots, \boldsymbol{v}_m' \}$ are linearly independent
  - Note notation $\boldsymbol{v} = v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n = \begin{bmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \boldsymbol{\mathscr{E}}\boldsymbol{v}$ where $\boldsymbol{\mathscr{E}} \in \mathcal{V}^n$, formalized as $\mathcal{V}^n \times {}^n\mathbb{R} \mapsto \mathcal{V}$
  - Proposition IV:
    - If $\boldsymbol{\mathscr{E}}\boldsymbol{v} = \boldsymbol{0}$ then $\boldsymbol{v} = \boldsymbol{0}$ because $\{ \boldsymbol{e}_1, \cdots, \boldsymbol{e}_m \}$ are linearly independent
    - If $\boldsymbol{\mathscr{E}}\boldsymbol{v} = \boldsymbol{\mathscr{E}}\boldsymbol{u}$ then $\boldsymbol{v} = \boldsymbol{u}$ since there is only one way to express a given vector as a linear combination of a set of independent vectors
  - Proof: $\sum_{j=1}^{n} \lambda_j \boldsymbol{v}_j = \boldsymbol{0} \iff \sum_{j=1}^{n} \lambda_j (\boldsymbol{\mathscr{E}}\boldsymbol{v}_j') = \boldsymbol{0} \iff \boldsymbol{\mathscr{E}} \left(\sum_{j=1}^{n} = \lambda_j \boldsymbol{v}_j' \right) = \boldsymbol{0} \iff \sum_{j=1}^{n} = \lambda_j \boldsymbol{v}_j' = \boldsymbol{0}$